MATHEMATICS

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MATHEMATICS 31

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MODULE 5

Curve Sketching

Distance Learning

Alberta EDUCATION

Mathematics 31

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CURVE SKETCHING

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Teachers (Mathematics 31)	✓
Administrators	
Parents	
General Public	
Other	

Mathematics 31
Student Module Booklet
Module 5
Curve Sketching
Alberta Distance Learning Centre
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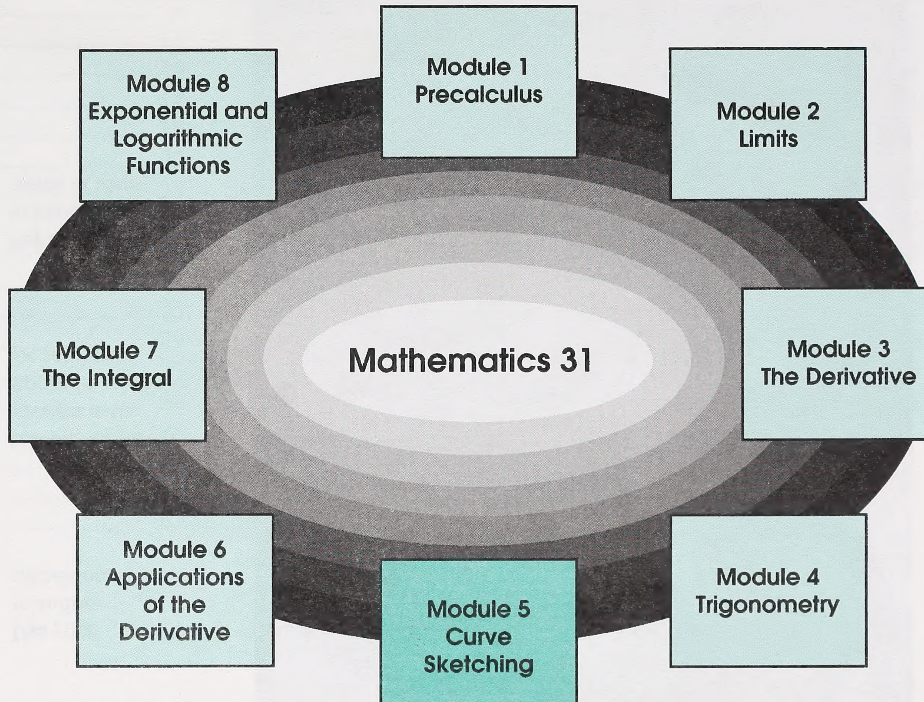
Welcome

WESTFILE INC.

Welcome to Module 5. We hope you'll enjoy your study of Curve Sketching.

Mathematics 31 contains eight modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



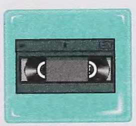
The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

- Use your scientific calculator.

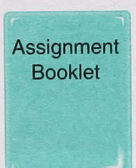
- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.



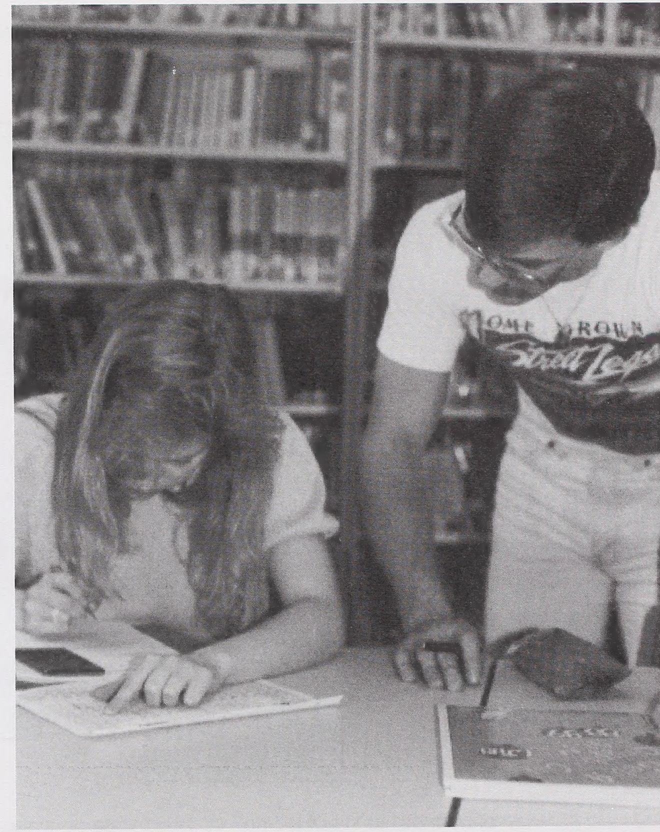
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

CONTENTS

CONTENTS (cont'd)

Module Overview

Have you ever walked along a beach and wished you could capture, on canvas, at least a fraction of the visual experience? It is the rare artist indeed that can recreate both a visual and emotional impression. In mathematics, graphing a function is much like being an artist. You are creating a visual representation of a function or relation. The graph must be accurate and complete.

This module develops procedures in curve sketching. You will use these procedures not only in calculus, but also in most branches of mathematics and the sciences. Sections 1 through 4 develop different aspects of these procedures.

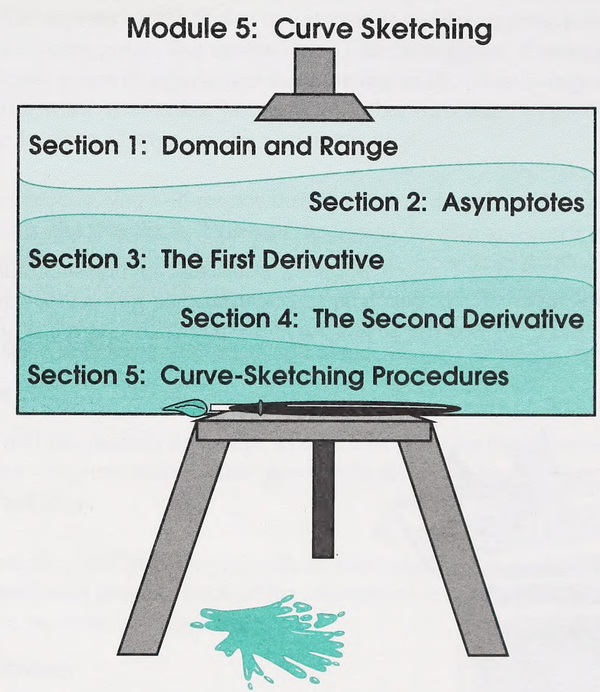
Section 1 reviews the concepts of domain and range, and how they can be used to locate the position of a function's graph in the plane.

Section 2 investigates curves which can be approximated by straight lines called asymptotes.

Section 3 extends the application of the first derivative in curve sketching. The first derivative is used to determine intervals on a graph that are rising and falling, and to locate horizontal and vertical tangents.

In Section 4, the second derivative is discussed in relation to the curvature of a graph. Points of transition in curvature, called inflection points, are also located.

In Section 5, a strategy for curve sketching that incorporates the techniques from this module and Module 1 is used. This graphing approach will be used to sketch algebraic and trigonometric functions.



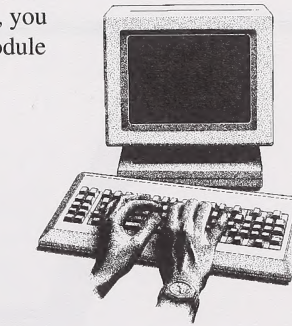
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete five section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**15 marks**
Section 2 Assignment	**15 marks**
Section 3 Assignment	**20 marks**
Section 4 Assignment	**15 marks**
Section 5 Assignment	**30 marks**
Final Module Assignment	**5 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

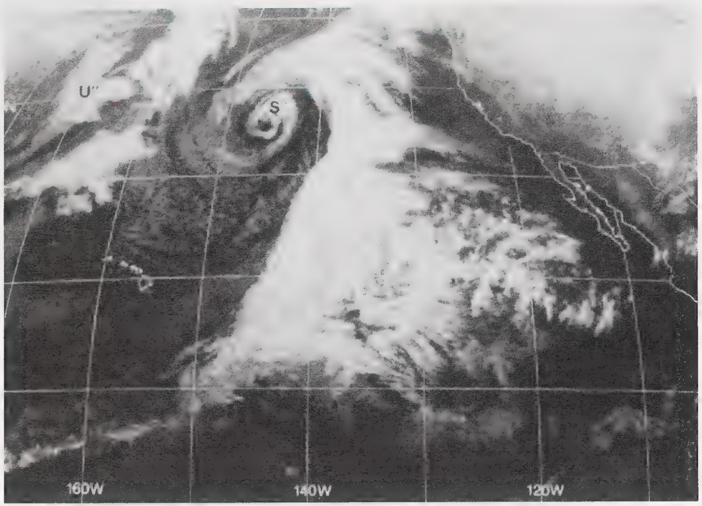
If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Domain and Range



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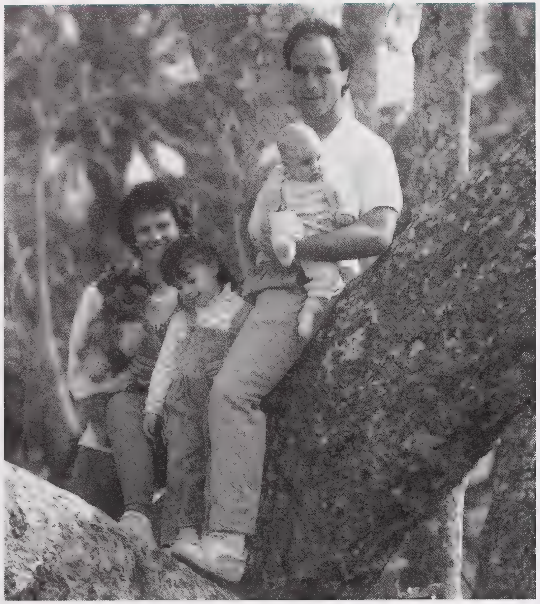
Using satellite photographs, the progress of weather systems off the coast of North America can be mapped. Weather forecasts for oceanic and onshore locations rely on accurate tracking of these systems. Longitude and latitude are used to pinpoint locations and describe regions on the globe; this system is similar to what you use to positioning points and curves in the Cartesian plane. Knowing where the graph of a particular function lies in the plane is important for your work in calculus, just as the weather forecaster's knowledge of the location of approaching systems is vital.

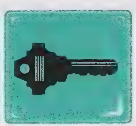
In this section, you will review the concepts of domain and range for relations and functions. You will determine domain and range of functions and relations from tables of values, from their graphs, and algebraically, from their equations. You will represent domain and range in interval notation. The x- and y-intercepts of the curves will assist you in determing domain and range and in positioning curves in the plane.

You will use domain and range to divide or partition the plane into regions—regions where either you will look for the graph or know it does not exist.

This section will provide you with an important tool in analysing and sketching graphs. Each of the subsequent sections will depend, in part, on what you learn here.

Activity 1: Domain and Range

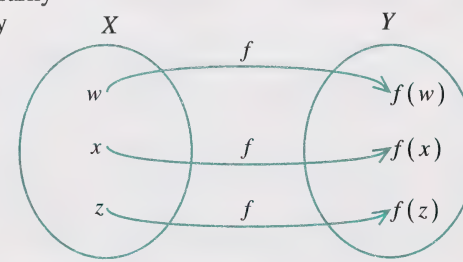




What do families have to do with domain and range? Recall that a **relation** in mathematics is a set of ordered pairs (x, y). If a particular ordered pair (x, y) is an element of relation R, then x is said to be related to y. The relation "is the mother of" consists of all ordered pairs (x, y), where x is the mother of y. The set of all x-values is the **domain** of the relation; here, it is the set of all mothers. The set of all y-values is the **range** of the relation; in this example, it is the set of all children—everyone!

Further, if, for each x-value in the domain, there is exactly one y-value in the range, the relation is called a **function**. The preceding relation is not a function! A given mother x may have more than one child y. However, the amount of income tax y you pay is a function of the amount x you earn. Based on your taxable income, there should only be one amount owed. You may write $y = f(x)$. The domain here is the set of all incomes, and the range of function f is the set of $f(x)$: the taxes corresponding to those incomes.

Most often, you have used the equation $y = f(x)$ to represent functions. For instance, $f(x) = x^2$. The values of f (or images under f) are found by substituting for x in the equation. Here, if $x = 7$, the value of f (or the image of 7 under f) is $f(7) = 7^2 = 49$. The association of values x of the domain X with values $f(x)$ of the range Y may be shown as in the diagram. Keep in mind that the sets X and Y are not necessarily disjoint sets; they may even be the same set!



Sometimes you will see the notation $f: x \rightarrow f(x)$ or $f: X \rightarrow Y$ or $X \xrightarrow{f} Y$ and will say ***f* maps *x* into $f(x)$** or ***f* maps *X* into *Y***.

Being able to determine the domain and range of a function or a relation is an essential skill in graphing.

Example 1

Determine the domain and range of $f(x)=3x-2$, where $-2 \le x < 3$. Express your answer in interval notation. Then, draw the graph.

Solution

In this example the domain is given. In interval notation, $-2 \le x < 3$ is $[-2, 3)$. This means $x \in [-2, 3)$.

Compare $f(x)=3x-2$ with $f(x)=mx+b$. Therefore, the slope $m=3$.

Since the function is linear, with a positive slope (when x increases, y increases), the least value of the function occurs when $x=-2$.

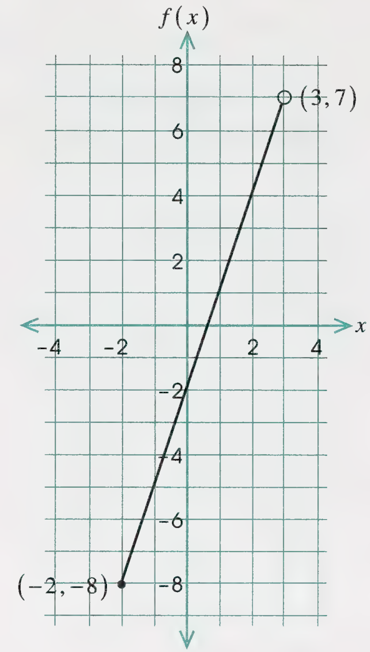
$$\begin{aligned} f(-2) &= 3(-2)-2 \\ &= -8 \end{aligned}$$

Also, the values of the function must be less than $f(3)$.

$$\begin{aligned} f(3) &= 3(3)-2 \\ &= 7 \end{aligned}$$

Each value of the function must lie in the interval $[-8, 7)$. This is the range of f. This means $f(x) \in [-8, 7)$ or $-8 \le f(x) < 7$.

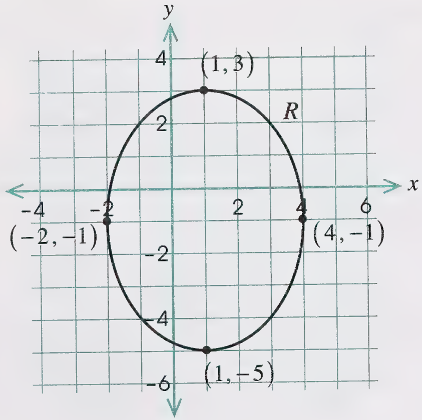
Notice that the point $(-2, -8)$ belongs to the graph; that fact is represented by a solid dot. The point $(3, 7)$ does not belong to the graph; that fact is represented by an open circle.



Domain and range can be determined from the graph of a relation or function.

Example 2

State the domain and range of the following relation R. The extreme points are given. Use interval notation.



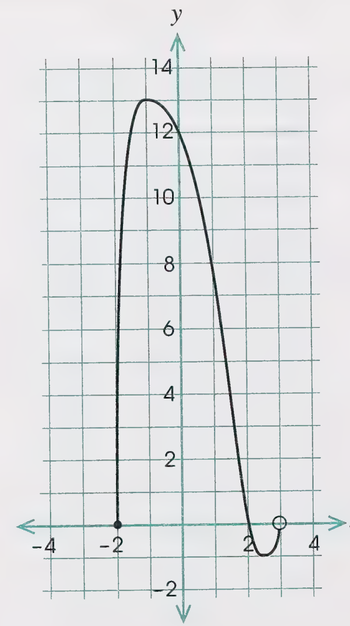
Solution

All points on the graph lie between $x = -2$ and $x = 4$, inclusive. These values are obtained directly from the x-axis. Therefore, the domain is $[-2, 4]$. All points on the graph lie between $y = -5$ and $y = 3$, inclusive. These values are obtained directly from the y-axis. Therefore, the range is $[-5, 3]$.

As you can see, the domain and range act as an address, telling the reader where the curve is located relative to the axes.

1. Find the domain and range of $f(x) = -x^2 + 2$, where $-2 \le x \le 3$; then graph the function. Express your answer using interval notation.

2. Find the domain and range for the following graph. Express your answer using interval notation.



Check your answers by turning to the Appendix.

In Example 3, you will see how the domain and range can be determined from the equation defining the relation or function.

Example 3

Determine the domain and range of each function. Confirm your results using a graphing calculator (or a computer program).

- $f(x)=\sqrt{x-3}$
- $f(x)=\dfrac{2x-3}{x+1}$
- $x^2-y^2=4$
- $y=2$
- $y=-\sqrt{25-x^2}$

Solution

$f(x)=\sqrt{x-3}$

Since the square roots of negative values are undefined, $x-3\geq 0$.

$$\begin{aligned} x-3&\geq 0\\ x&\geq 3 \end{aligned}$$

Therefore, the domain is $[3,\infty)$.

Since $\sqrt{x-3}$ is a principal or non-negative root, $y\geq 0$.

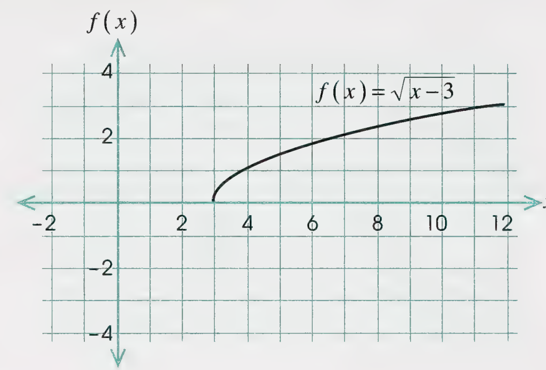
Therefore, the range is $[0,\infty)$.

To check these results on a graphing calculator, enter the function.

Graph $\sqrt{\ }$ (X,θ,T − 3) EXE

Note: You may need to check the calculator's owner's manual for the required sequence.

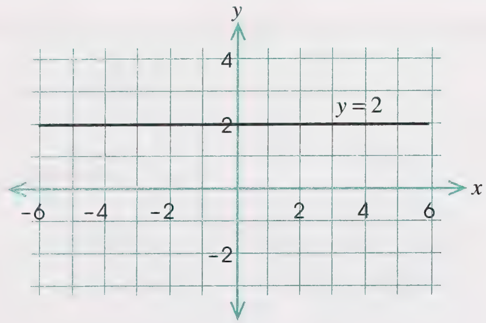
Notice that the graph extends to the upper right from $(3,0)$.



$y=2$

$y=2$ is a constant function. For any real value of x, the value of y is 2. That is, $f(x)=2$ for all $x\in R$.

The domain is the set of reals; the range is $\{2\}$.



$$f(x) = \frac{\mathbf{2x-3}}{\mathbf{x+1}}$$

Since division by 0 is undefined, $x \neq -1$. This means $x < -1$ or $x > -1$. In interval notation, the domain is $(-\infty, -1) \cup (-1, \infty)$.

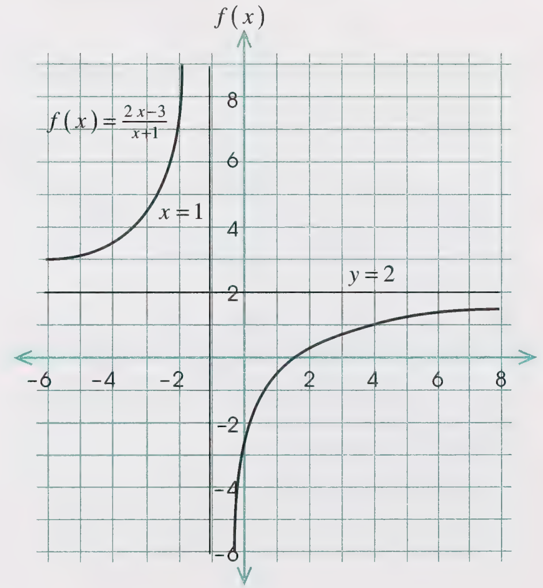
To find the range, solve the equation for *x* as subject; that is, $x = g(y)$.

$$\begin{aligned} y &= \frac{2x-3}{x+1} \\ y(x+1) &= 2x-3 \\ xy + y &= 2x-3 \\ xy - 2x &= -y-3 \\ x(y-2) &= -y-3 \\ x &= \frac{-y-3}{y-2} \end{aligned}$$

The fractional expression is undefined when $y = 2$.

Therefore, the range is $\{f(x) \mid f(x) \neq 2\}$ or $(-\infty, 2) \cup (2, \infty)$.

Notice that the graph does not intersect $x = -1$ or $y = 2$.



$y = -\sqrt{25 - x^2}$

Since the square roots of negative values are non-real,

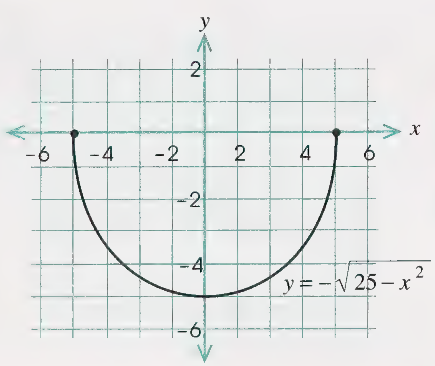
$$\begin{aligned} 25 - x^2 &\geq 0 \\ x^2 &\leq 25 \\ -5 \leq x &\leq 5 \end{aligned}$$

The domain is $[-5, 5]$.

Since $y = -\sqrt{25 - x^2}$, y is non-positive. Also, the maximum value of $25 - x^2$ occurs when $x = 0$. Now, when $x = 0$, $y = -\sqrt{25 - 0^2} = -5$.

Therefore, range is $-5 \leq y \leq 0$ or $[-5, 0]$.

Notice that the graph is a semicircle; relative to the axes, the graph lies between $x = -5$ and $x = 5$ on the one hand, and $y = 0$ and $y = -5$ on the other.



$x^2 - y^2 = 4$

If you rewrite the relation as one or more explicit functions of x, restrictions on x become more obvious.

$$\begin{aligned} x^2 - y^2 &= 4 \\ y^2 &= x^2 - 4 \\ y &= \pm\sqrt{x^2 - 4} \end{aligned}$$

Now, $x^2 - 4$ must be non-negative!

$$\begin{aligned} x^2 - 4 &\geq 0 \\ x^2 &\geq 4 \\ \therefore\ x \geq 2 &\text{ or } x \leq -2 \end{aligned}$$

The domain is $[2, \infty) \cup (-\infty, -2]$.

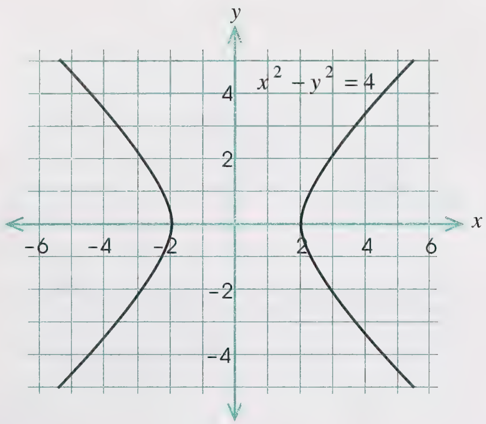
When $x = \pm 2$, $y = 0$. When $x \geq 2$ or $x \leq -2$, y is either positive or negative without bound. Therefore, the range is the set of reals.

To graph this directly on a graphing calculator, you must either enter both $y = \sqrt{x^2 - 4}$ and $y = -\sqrt{x^2 - 4}$ or use the conics program you may have stored from your study of quadratic relations in Mathematics 30.

Graph $\sqrt{\ }$ (X,θ,T Shift $\sqrt{\ }$ – 4) EXE

Graph – $\sqrt{\ }$ (X,θ,T Shift $\sqrt{\ }$ – 4) EXE

Compare your display on the graphing calculator with the following graph. Notice that the graph consists of two branches, one opening to the right from $(2, 0)$, and one opening to the left from $(-2, 0)$. The graph extends infinitely, upward and downward.



Note: A computer graphing program yields more detail than a graphing calculator.

Example 4

Find the domain and range of $y = \frac{1}{\sqrt{x^2 - 3x - 4}}$.

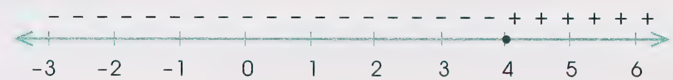
Solution

First, $x^2 - 3x - 4$ cannot be 0, since division by 0 is undefined.

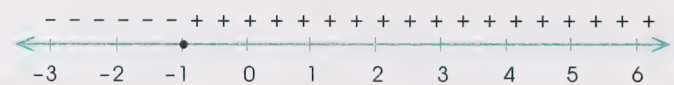
$$x^2 - 3x - 4 \neq 0$$
$$(x-4)(x+1) \neq 0$$
$$x - 4 \neq 0 \quad \text{and} \quad x + 1 \neq 0$$
$$x \neq 4 \qquad\qquad x \neq -1$$

Also, $x^2 - 3x - 4$ must be positive.

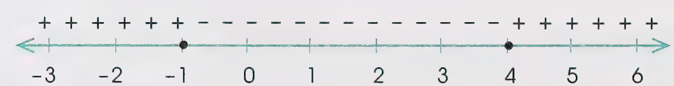
Sign of $(x-4)$



Sign of $(x+1)$



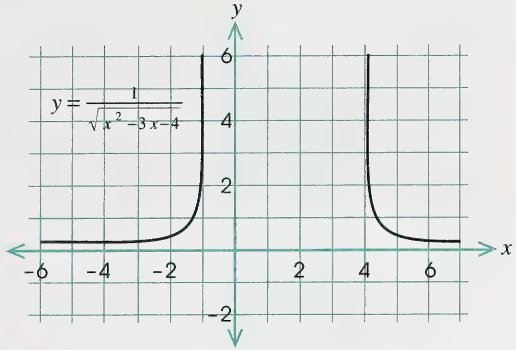
Sign of $(x-4)(x+1)$



The domain is $(-\infty, -1) \cup (4, \infty)$.

The range is $(0, \infty)$. Since the denominator of the function is positive, the values of the function are positive. As the denominator $\to 0$, the numerator $\to \infty$. As the denominator $\to \infty$, the function $\to 0$.

Verify the following graph using a graphing calculator.



Use a graphing calculator (or computer program) to verify question 3.

3. Find the domain and range of each function or relation from its equation, unless stated otherwise.

a. $f(x) = -|x-2|+1$

b. $f(x) = \frac{2x-3}{x+3}$

c. $f(x) = \sqrt{x^2 - 9}$

d. $f(x) = \sqrt{-x^2 + 5x - 4}$ (Find the domain only.)

e. $x^2 + 4y^2 = 4$

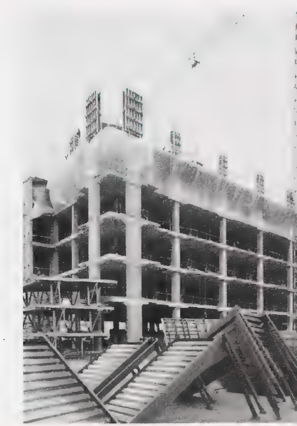
f. $f(x) = \frac{|x|}{x}$

Check your answers by turning to the Appendix.

A function in mathematics describes a relationship between two variables. Domain and range refer to the sets of permissible values of each variable. Family relationships are similar; the concept of "niece" composes two sets of people—aunts and uncles with nieces.

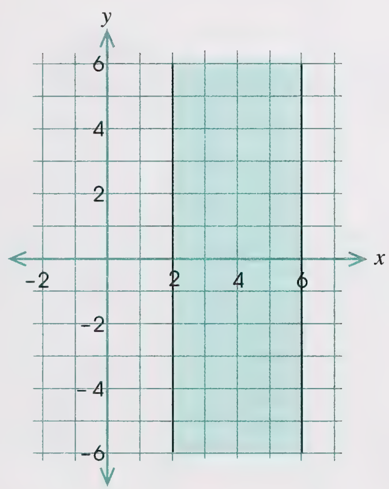
Activity 2: Partitioning the Plane

The building in the photograph will contain a number of offices separated from each other by concrete partitions. In mathematics, the graph of a given function or relation lies in a particular "office" in the plane. The domain and range determine the location of the partitions defining that office.

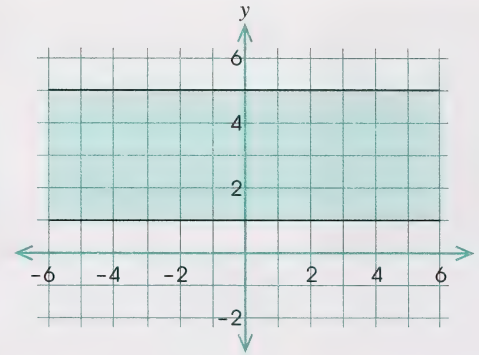


As you saw in the previous activity, domain and range serve as an address, indicating the location of the relation's or function's graph relative to the axes. You can also think of the domain and range as dividing or partitioning the plane into regions: regions that either contain the graph or do not. Just as when building a home, you partition the living area into separate rooms with each room serving as a specific function, so, too, does the domain and range partition the plane.

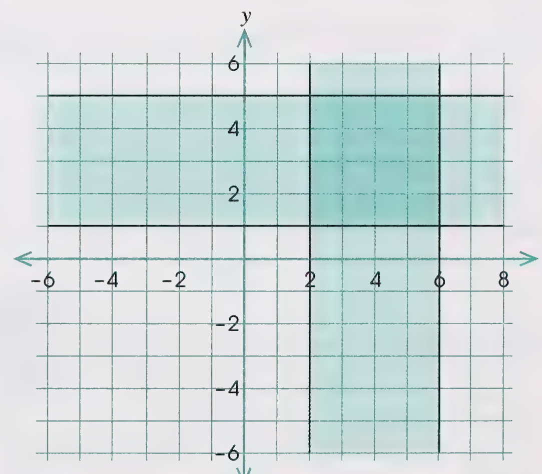
If, for instance, the domain of a relation were $[2, 6]$, its graph would lie in the region of the plane between the vertical lines $x = 2$ and $x = 6$.



Similarly, if the range of the relation were $[1, 5]$, the graph would lie in the region between the horizontal lines $y = 1$ and $y = 5$.

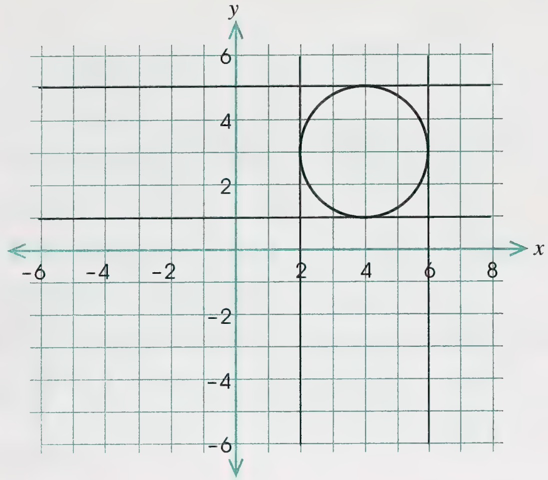


The graph, therefore, would lie in the darker shaded region shown in the following grid.



An example of a relation in this region is $(x-4)^2+(y-3)^2=2^2$.

This is a circle centred at the point $(4, 3)$ and with radius 2.



Example 1

Find the domain and range of $y=\sqrt{x^2-9}+2$.
Illustrate the region of the plane containing the graph.
Sketch the graph using a graphing calculator (or computer program).

Solution

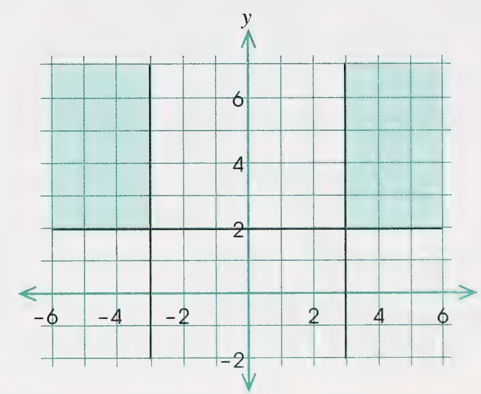
The domain of the function consists of those values of x for which x^2-9 is non-negative.

$$x^2-9\geq 0$$
$$x^2\geq 9$$
$$\therefore\ x\geq 3 \text{ or } x\leq -3$$

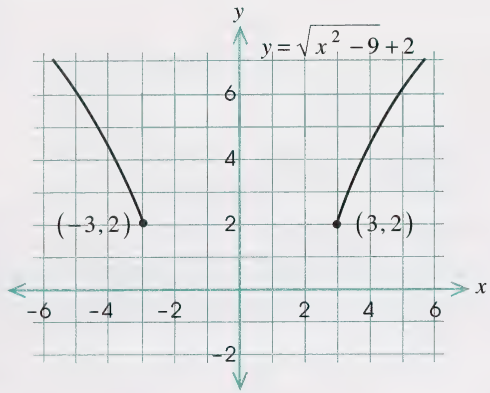
The domain is $(-\infty, -3]\cup[3, \infty)$.

Since $\sqrt{x^2-9}$ is a principal or non-negative root, $\sqrt{x^2-9}+2\geq 0+2$. Therefore, $y\geq 2$.

The graph must lie in the shaded regions of the plane.



This is confirmed from the graph.



Example 2

Find the domain and range of $y=-\frac{1}{\sqrt{x^2-x-6}}+x$.

Illustrate the region of the plane containing the graph. Sketch the graph using a graphing calculator (or computer program).

Solution

The domain of the function consists of only those values of x for which x^2-x-6 is positive. The function x^2-x-6 cannot be zero since y would be undefined.

$$x^2-x-6>0$$
$$(x+2)(x-3)>0$$

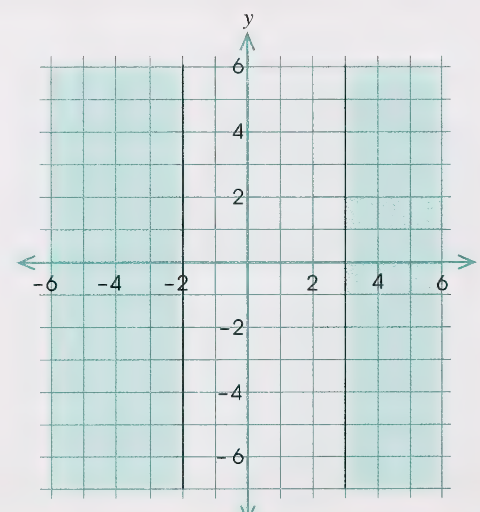
If $(x+2)(x-3)>0$, then both factors are positive or both factors are negative.

If $x+2>0$ and $x-3>0$, then $x>3$.
If $x+2<0$ and $x-3<0$, then $x<-2$.

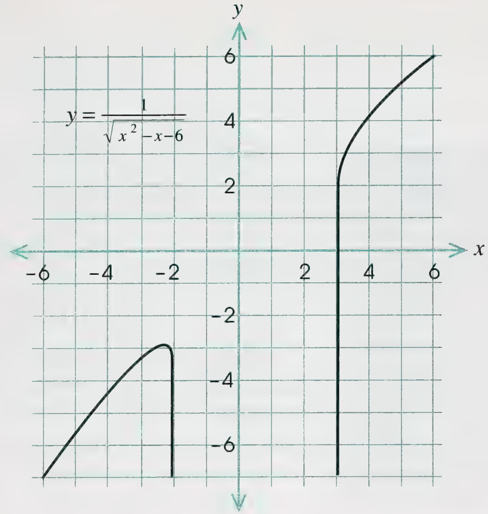
Therefore, the domain is $(-\infty,-2)\cup(3,\infty)$.

The range is the set of reals.

Therefore, the graph lies in the shaded regions of the plane.

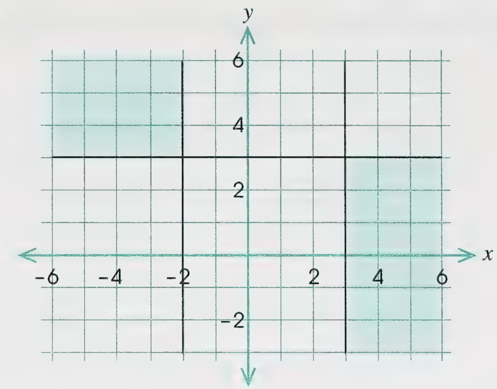


This is confirmed by the graph.

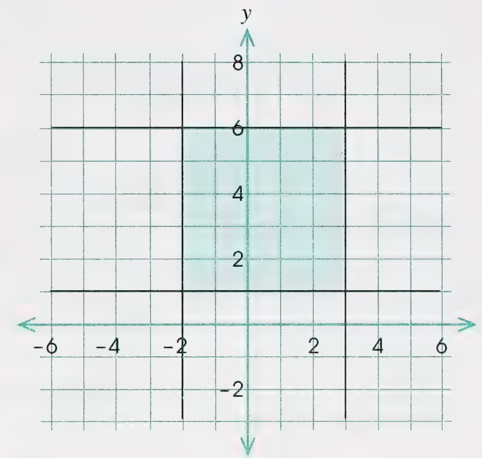


1. What would the domain and range be of the relation that lies in the shaded regions of the following grids? Assume that the relation is defined for values of x or y on the boundary lines.

a.



b.



2. Give an example of a function with a graph lying strictly above the x-axis.

Use a graphing calculator to verify the answers to question 3.

3. Determine the domain and range of each function or relation; then sketch each graph.

a. $y=-x^{\frac{2}{3}}$ **b.** $y=\left|x^2-2\right|$ **c.** $y=\dfrac{|x+2|}{x}$

Check your answers by turning to the Appendix.

You should now be able to use domain and range as a method of partitioning the plane. Think of the partitioned regions as separate rooms or offices of an office building. Some are empty, and others contain the graph of the function under considerations.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

The domain and range of the standard conics and other relations, if they are centred at the origin, can easily be determined using intercepts. Once the intercepts are located algebraically, the curve may be quickly sketched. The domain and range may then be read directly from the graph.

Example 1

Determine the domain and range of the circle $x^2+y^2=16$.

Solution

To find the x-intercepts, replace y by 0.

$$\begin{aligned} x^2+0^2&=16\\ x^2&=16\\ x&=\pm 4\end{aligned}$$

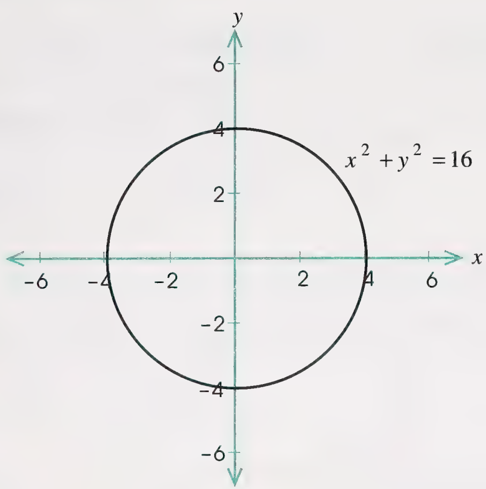
The graph intersects the x-axis at $(\pm 4, 0)$.

To find the y-intercepts, replace x by 0.

$$0^2 + y^2 = 16$$
$$y^2 = 16$$
$$y = \pm 4$$

The graph intersects the y-axis at $(0, \pm 4)$.

As expected, the x- and y-intercepts are the same. Sketch the graph using these values.



Therefore, the domain is $[-4, 4]$, and the range is $[-4, 4]$.

Example 2

Find the domain and range of the ellipse $4x^2 + 25y^2 = 100$.

Solution

Find the x-intercepts.

Let $y = 0$.

$$4x^2 + 25(0)^2 = 100$$
$$4x^2 = 100$$
$$x^2 = 25$$
$$x = \pm 5$$

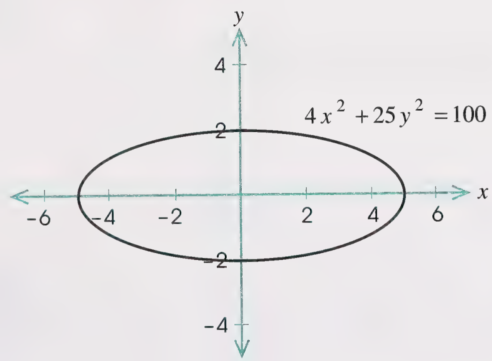
The ellipse crosses the x-axis at $(\pm 5, 0)$.

Find the y-intercepts.

Let $x = 0$.

$$4(0)^2 + 25y^2 = 100$$
$$25y^2 = 100$$
$$y^2 = 4$$
$$y = \pm 2$$

The ellipse crosses the y-axis at $(0, \pm 2)$. Sketch the curve from its intercepts.



The domain is $[-5, 5]$ and the range is $[-2, 2]$.

Example 3

Find the domain and range of the hyperbola $x^2 - 4y^2 = 8$.

Solution

Find the x-intercepts.

When $y = 0$, $x^2 - 4(0)^2 = 8$

$$x^2 = 8$$

$$x = \pm 2\sqrt{2}$$

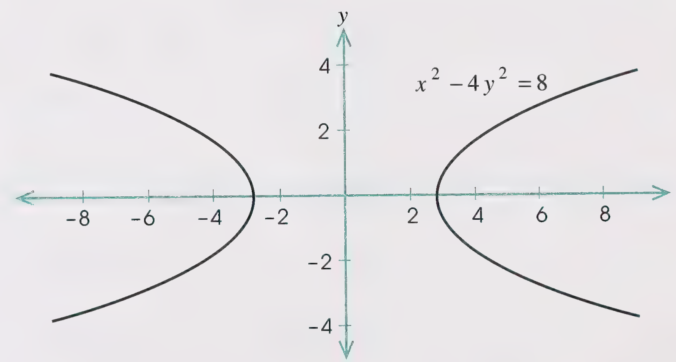
The hyperbola crosses the x-axis at $(\pm 2\sqrt{2}, 0)$.

When $x = 0$, $0^2 - 4y^2 = 8$

$$y^2 = -2 \quad \text{(Impossible!)}$$

The graph of the hyperbola does cross the x-axis. The branches of the curve open to the left and right.

The domain is $(-\infty, -2\sqrt{2}] \cup [2\sqrt{2}, \infty)$. Now, sketch the graph.



Because the graph extends infinitely upward and downward, the range is the set of reals.

1. Find the domain and range of $x^2 + y^2 = 1$. Sketch the graph.

2. Find the domain and range of $2x^2 + 5y^2 = 40$. Sketch the graph.

3. Find the domain and range of $x^2 - y^2 + 1 = 0$. Sketch the graph.

4. Find the domain and range of $3x^2 - y^2 = 1$. Sketch the graph.

Check your answers by turning to the Appendix.

Enrichment

As you have seen, you can discover a great deal about the graph of a relation from its domain and range. Even more can be discovered if the facts about domain and range are combined with knowledge of symmetry, intercepts, and the slope at the intercepts.

Example

Discuss the graph of $y^2 = x^2\left(4 - x^2\right)$. Mention symmetry, intercepts, slope at the intercepts, and domain and range. Use this information to sketch the graph.

Solution

The graph of a relation is symmetric with respect to the *x*-axis if its equation remains unchanged when (x, y) is replaced by $(x, -y)$. Since $(-y)^2 = y^2 = x^2\left(4 - x^2\right)$, this graph is symmetric about the *x*-axis.

Similarly, the graph of a relation is symmetric with respect to the *y*-axis if the equation remains unchanged when (x, y) is replaced by $(-x, y)$. Here, $y^2 = (-x)^2\left[4 - (-x)^2\right]$ is identical to $y^2 = x^2\left(4 - x^2\right)$. Therefore, the graph is symmetric about the *y*-axis. Furthermore, the graph is also symmetric with respect to the origin, since replacing (x, y) by $(-x, -y)$ does not change the form of the equation.

$$(-y)^2 = (-x)^2\left[4 - (-x)^2\right]$$
$$y^2 = x^2\left(4 - x^2\right)$$

Now, find the *x*-intercepts.

$$x^2\left(4 - x^2\right) = 0$$

$$\begin{array}{lll} x^2 = 0 & \text{or} & 4 - x^2 = 0 \\ x = 0 & & x^2 = 4 \\ & & x = \pm 2 \end{array}$$

Therefore, the graph crosses the *x*-axis at $(0, 0)$, $(2, 0)$, and $(-2, 0)$.

Now find the y-intercept.

$$y^2 = (0)^2\left[4-(0)^2\right]$$
$$y = 0$$

The graph only crosses the y-axis at the origin.

Next, what are the domain and range?

Since the left side of the equation is y^2, the left side is non-negative. Therefore, on the right side, $x^2\left(4-x^2\right)$, the factor $4-x^2$ must also be non-negative.

$$\begin{aligned} 4-x^2 &\geq 0 \\ x^2 &\leq 4 \\ -2 \leq x &\leq 2 \end{aligned}$$

The domain is $[-2, 2]$.

What is the range? Investigate the equation of the relation more closely.

$$\begin{aligned} y^2 &= x^2\left(4-x^2\right) \\ y^2 &= 4x^2 - x^4 \\ x^4 - 4x^2 + y^2 &= 0 \end{aligned}$$

This equation is of the quadratic type. If it is rewritten as $\left(x^2\right)^2 - 4\left(x^2\right) + y^2 = 0$ and compared to $ax^2+bx+c=0$, then use the quadratic formula.

$$\begin{aligned} x^2 &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \text{ where } a=1,\ b=-4, \text{ and } c=y^2 \\ &= \frac{4 \pm \sqrt{(-4)^2-4(1)y^2}}{2} \\ &= \frac{4 \pm \sqrt{16-4y^2}}{2} \end{aligned}$$

This tells you $16-4y^2 \geq 0$

$$\begin{aligned} 16 &\geq 4y^2 \\ y^2 &\leq 4 \\ -2 \leq y &\leq 2 \end{aligned}$$

The range is $[-2, 2]$.

Out of curiosity, what is x when $y = \pm 2$?

$$\therefore\ x^2 = \frac{4 \pm \sqrt{16-4(\pm 2)^2}}{2}$$

$$x^2 = \frac{4 \pm \sqrt{0}}{2}$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

The highest points on the graph are $\left(\pm\sqrt{2}, 2\right)$; the lowest points are $\left(\pm\sqrt{2}, -2\right)$.

Finally, what are slopes of the graph at the intercepts?

Differentiate implicitly the following form of the relation.

$$x^4 - 4x^2 + y^2 = 0$$

$$4x^3 - 8x + 2yy' = 0$$

$$2x^3 - 4x + yy' = 0$$

$$yy' = 4x - 2x^3$$

$$y' = \frac{4x - 2x^3}{y}$$

At $(2,0)$, $y' = \dfrac{4(2) - 2(2)^3}{0}$

$= \text{undefined}$

The tangent at $(2,0)$ is vertical. Similarly, the tangent at $(-2,0)$ is vertical. Now what about $(0,0)$? At the origin $y' = \frac{0}{0}$, which is meaningless. However, if you solve the original equation for *y*, and substitute into the derivative y', finite values of the slope occur!

First solve for *y*.

$$y^2 = x^2\left(4 - x^2\right)$$

$$y = \pm x\sqrt{4 - x^2}$$

Substitute in the expression for y'.

$$y' = \frac{4x - 2x^3}{\pm x\sqrt{4 - x^2}}$$

$$= \frac{x\left(4 - 2x^2\right)}{\pm x\sqrt{4 - x^2}}$$

$$= \frac{4 - 2x^2}{\pm\sqrt{4 - x^2}}$$

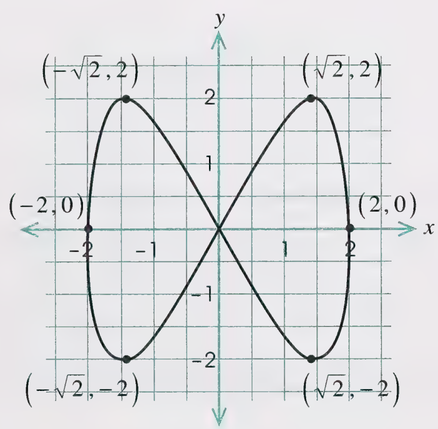
At $(0,0)$, $y' = \dfrac{4 - 2(0)^2}{\pm\sqrt{4 - (0)^2}}$

$= \pm 2$

Apparently, the curve crosses itself at the origin, and the two tangents have slopes of ± 2.

In summary, this relation is symmetric with respect to both axes and the origin. Its graph lies entirely within a square located at $-2 \le x \le 2$ and $-2 \le y \le 2$. It has minimum and maximum points at $\left(\pm\sqrt{2}, -2\right)$ and $\left(\pm\sqrt{2}, 2\right)$. At $(\pm 2, 0)$ the tangents are vertical; at the origin the tangents intersect and have slopes ± 2.

Graph the relation using these facts!



1. Given $y^2 = x(x-4)$, do the following:

   a. Investigate symmetry.
   b. State the domain and range.
   c. Determine the intercepts.
   d. State the slope of the curve at its intercepts.
   e. Graph the relation.

2. Describe the effect on the graph in question 1 if the equation were rewritten as $x^2 = y(y-4)$.

Check your answers by turning to the Appendix.

Conclusion

In this section, you reviewed the concepts of domain and range for relations and functions. You should be able to determine, and write using interval notation, the domain and range of functions and relations from tables of values, from their graphs, and algebraically from their equations. You should be able to use the *x*- and *y*-intercepts of the curves to assist you in positioning those curves in the plane.

You also saw how domain and range are used to divide or partition the plane into regions—regions where either you look for the graph or know it does not exist.

This section provided you with an important tool in analysing and sketching graphs. What you studied here you will use in each of the subsequent sections.

Analysing graphs involves many of the same skills as map reading. A meteorologist, plotting the progress of a storm system, uses map coordinates just as you would use the coordinate axes to locate a curve in the plane. The extent of the storm can be described in terms similar to those you used to describe the domain and range of a relation. The terminology may differ, but the fundamentals are still the same.

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Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 2: Asymptotes


NASA

Many people dream of travelling to the stars. Unfortunately, distant galaxies, many light years away, may always be inaccessible. Relativity theory states that the speed of light is an absolute; that nothing can travel faster. Functions that model what would happen as a spacecraft approaches that speed indicate that the mass of the craft increases without bound.

How the graph of a function behaves as either the dependent or independent variable becomes infinite is an important consideration in the analysis of that function, particularly if that function models a real-world situation.

In this section you will investigate curves that approach lines, but never touch, the farther you move away from the origin. These lines are called asymptotes. You will investigate conditions for which asymptotes occur, and you will practise procedures for finding asymptotes.

Activity 1 deals specifically with vertical asymptotes. Activity 2 reviews the techniques for finding horizontal asymptotes, which were introduced in the discussion of limits in Module 2. Activity 3 deals with oblique asymptotes—asymptotes that are neither horizontal nor vertical.

Activity 1: Horizontal Asymptotes

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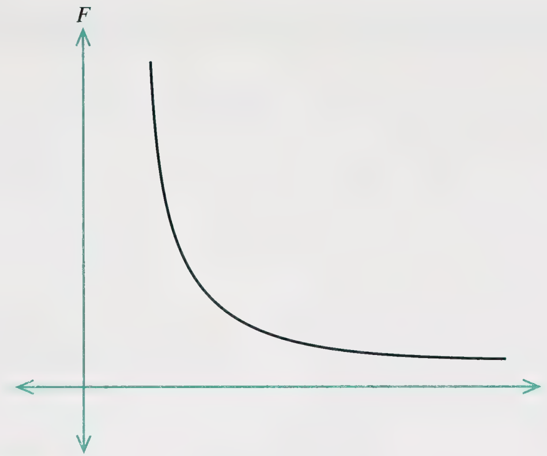
All objects in the universe interact. The gravitational force F between two masses at rest is given by $F = \frac{C}{r^2}$, where r is the distance between the two masses and C is a constant. This is called an **inverse-square** law. If the distance is doubled, the force decreases by a factor of 4. As the distance between the masses increases, the gravitational force between them decreases, approaching 0. But, is the force between any two masses in the universe ever zero? The following graph illustrates this relationship.



In this situation, the horizontal axis is an **asymptote**. Recall from Module 2 that an asymptote is a line a curve approaches, but never touches, as you move farther and farther away from the origin, along that curve. The distance between the curve and the asymptote tends to zero, as the distance between a point on the curve and the origin becomes infinite.

Also, recall that horizontal asymptotes of a function $y = f(x)$ are found by finding limits as $x \to \pm\infty$.

Example 1

Detemine the horizontal asymptote(s) of $f(x) = \frac{3x-2}{x-1}$.

Solution

Use a calculator to establish a table of values to investigate what happens as $x \to \pm\infty$. Round all values to five decimal places.

x	$f(x)$
10	3.111 11
100	3.010 10
1000	3.001 00
10 000	3.000 10

x	$f(x)$
−10	2.909 09
−100	2.990 10
−1000	2.999 01
−10 000	2.999 90

It appears that whether x approaches $+\infty$ or $-\infty$, $f(x) \to 3$. Using limits, how can you prove that the horizontal asymptote is $y = 3$?

$$\lim_{x\to\infty} \frac{3x-2}{x-1} = \lim_{x\to\infty} \frac{x\left(3-\frac{2}{x}\right)}{x\left(1-\frac{1}{x}\right)}$$

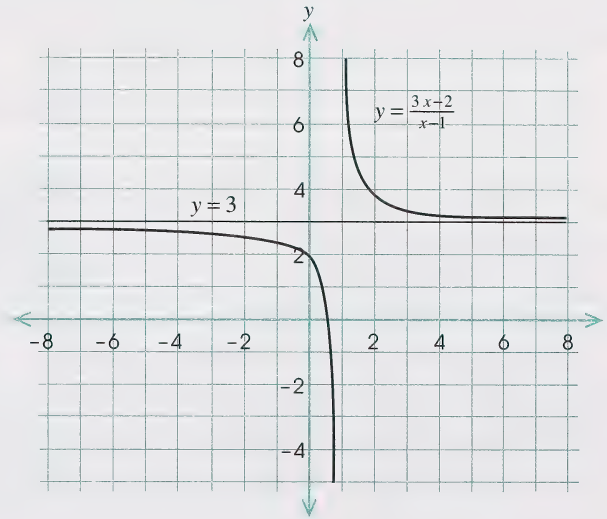
Remove the highest power of x as a common factor.

$$= \frac{3-0}{1-0}$$
$$= 3$$

$\frac{2}{x}$ and $\frac{1}{x}$ tend to 0 as $x \to \infty$.

The horizontal asymptote is $y = 3$.

The graph confirms this result.



Example 2

Determine the asymptote(s) of $y = \frac{3-3x^2}{x^2+2}$; then, sketch the graph of the function.

Solution

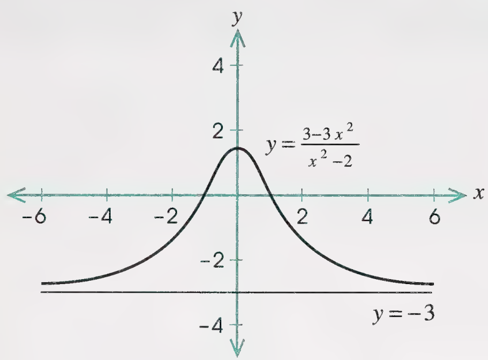
$$\lim_{x\to\infty}\frac{3-3x^2}{x^2+2}=\lim_{x\to\infty}\frac{x^2\left(\frac{3}{x^2}-3\right)}{x^2\left(1+\frac{2}{x^2}\right)}$$

Remove x^2 as a common factor.

$$=\frac{0-3}{1+0}$$
$$=-3$$

The horizontal asymptote is $y=-3$.

The graph of the function is as follows:



Example 3

Does $y=2^x-4$ have a horizontal asymptote? Sketch the graph.

Solution

When $x\to+\infty$, $y\to+\infty$ as well. There appears to be no horizontal asymptote.

However, if $x\to-\infty$, then $y\to-4$, since $2^x\to 0$.

Consider the following table of values.

x	y
−1	−3.5
−2	−3.75
−3	−3.875
−4	−3.9375
⋮	⋮
−10	−3.9990

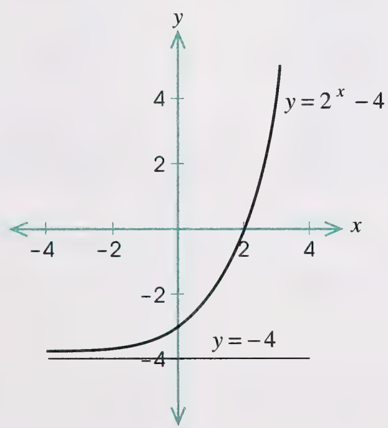
The table implies that the horizontal asymptote is $y=-4$.

Algebraically, the limit is determined as follows:

$$\lim_{x \to -\infty} \left(2^x - 4\right) = 0 - 4$$
$$= -4$$

$2^x \to 0$

These results are more apparent from the graph.



The preceding example illustrates that, for certain functions such as exponential functions, the limit as $x \to +\infty$ must be checked separately from the limit as $x \to -\infty$.

Use a graphing calculator (or computer program) to check your answers for questions 1 and 2.

1. Find the horizontal asymptote(s), if any, for each function.

   a. $y = -x^3 + x$ **b.** $y = \dfrac{2}{x} + 3$ **c.** $y = \dfrac{2x^2 - 1}{x + 1}$

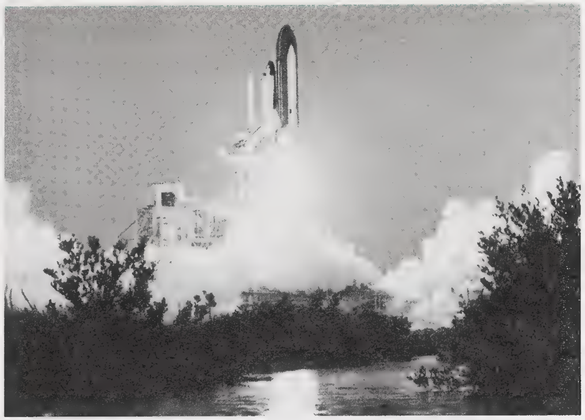
2. Find the horizontal asymptote(s), if any, for each function.

   a. $y = \dfrac{x^2}{x^2 + 1}$ **b.** $y^2(x - 1) = x$ **c.** $y = -2^{-x} + 1$

Check your answers by turning to the Appendix.

The inverse-square law describing the gravitational force between two masses is only one example of a function with a horizontal asymptote. Can you think of any other relationships from the physical sciences for which this situation is true?

Activity 2: Vertical Asymptotes

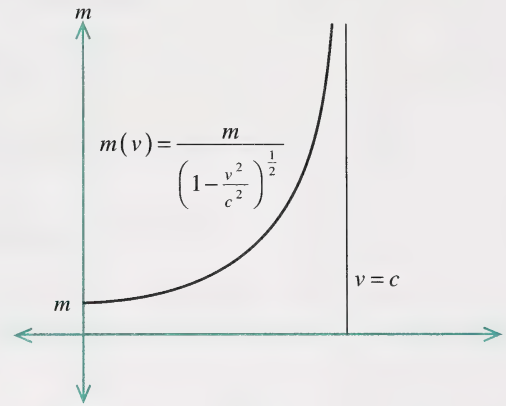


NASA

Have you ever dreamed of travelling to the stars? The vast distances involved could only be bridged by spacecraft travelling at tremendous speeds. In relativity theory, the speed of light is a limiting factor. According to this theory, which has been verified by various electron deflection experiments, the mass of an object $m(v)$ increases dramatically as its speed v approaches the speed of light c.

The relationship is $m(v) = \dfrac{m}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}$, where m is the rest mass of the object.

As the object's speed v approaches c, the fraction $\frac{v^2}{c^2}$ approaches 1, and the denominator of the formula approaches 0. The relativistic mass of a spaceship becomes infinite. The forces required to propel a spaceship close to the speed of light become prohibitive! The graph of the relationship between mass and speed is shown here.



The vertical line that the curve approaches as $m(v)$ becomes infinite is a **vertical asymptote**.

The preceding discussion illustrates the method for finding vertical asymptotes. If the curve has an equation of the form $f(x) = \frac{P(x)}{Q(x)}$, where the numerator and denominator have no common factors, and $x = a$ makes the denominator 0 (but not the numerator), then the distance a point on the curve lies from the origin becomes infinte as $x \to a$. Therefore, $x = a$ is a vertical asymptote of the curve.

Example 1

Find the vertical asymptotes of $y = \frac{1}{x-2}$. Sketch the graph of the function.

Solution

The denominator $x - 2$ is zero at $x = 2$.

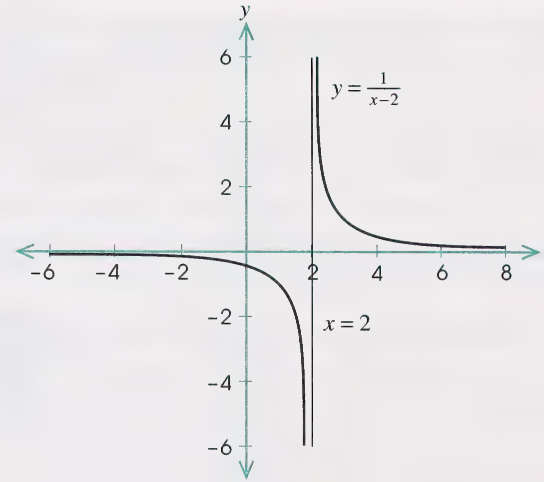
Investigate what happens as $x \to 2^-$ and $x \to 2^+$.

When x approaches 2 from the left, $y \to -\infty$. When x approaches 2 from the right, $y \to +\infty$. Look at the following tables of values.

x	y
1.9	−10
1.99	−100
1.999	−1000
1.9999	−10 000

x	y
2.1	10
2.01	100
2.001	1000
2.0001	10 000

The vertical asymptote is $x = 2$. Notice that this function has a horizontal asymptote as well, namely $y = 0$.



The following example illustrates what can occur when the numerator and denominator have a factor in common.

Example 2

Find the vertical asymptote(s) of $y = \frac{x-2}{x^2 - 3x + 2}$.

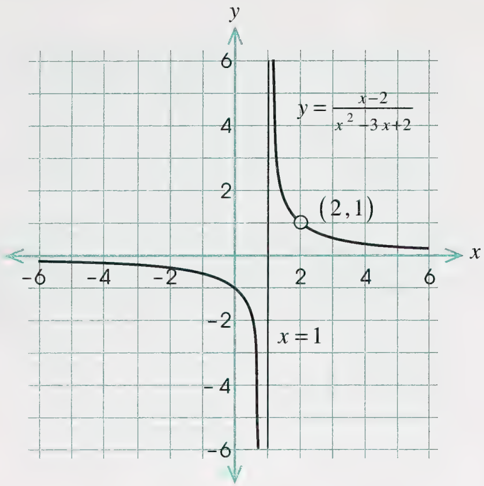
Solution

Factor the denominator.

$$y = \frac{x-2}{(x-2)(x-1)}$$

Since the denominator is 0 when $x = 2$ and when $x = 1$, you might be tempted to say there are two asymptotes. Don't you dare! The fraction can be reduced to $y = \frac{1}{x-1}$ as long as $x \neq 2$. The point $(2, 1)$ is simply a discontinuity on the curve and is indicated by the open circle. The only vertical asymptote is $x = 1$. The horizontal asymptote is $y = 0$.

The following graph illustrates this.



In the next example, there are two vertical asymptotes.

Example 3

Find the vertical asymptotes of $y = \dfrac{x^3}{x^2 - x - 12}$.

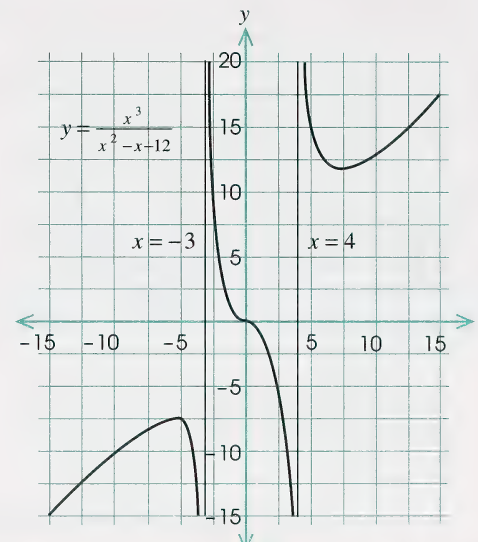
Solution

Factor the denominator.

$$y = \frac{x^3}{(x-4)(x+3)}$$

The vertical asymptotes are $x = -3$ and $x = 4$.

The graph confirms these asymptotes.



Use a graphing calculator (or computer program) to check your answers to question 1.

1. Sketch the graphs of the following. Show both horizontal and vertical asymptotes.

a. $y=\dfrac{1}{(1-x)^{\frac{1}{2}}}$

b. $y=\dfrac{4-x}{3x+8}$

c. $y=\dfrac{x^3-4x}{x^2-9}$

d. $y=\dfrac{x-4}{x^2-4x}$

Check your answers by turning to the Appendix.

Example 4

Find the asymptotes of $y^2\left(x^2-x\right)=x^2+4$. Discuss symmetry, domain, and range; then, sketch the graph.

Solution

Find the vertical asymptotes. Factor the denominator.

$$\begin{aligned} y^2 &= \frac{x^2+4}{x^2-x} \\ &= \frac{x^2+4}{x(x-1)} \end{aligned}$$

The vertical asymptotes are $x=0$ and $x=1$.

Find the horizontal asymptotes. Determine the limit as $x\to\infty$.

$$\begin{aligned} \lim_{x\to\infty}\frac{x^2+4}{x^2-x} &= \lim_{x\to\infty}\frac{x^2\left(1+\frac{4}{x^2}\right)}{x^2\left(1-\frac{1}{x}\right)} \\ &= \frac{1+0}{1-0} \\ &= 1 \end{aligned}$$

$\therefore\ y^2=1$

The horizontal asymptotes are $y=\pm 1$.

The fact that there are two horizontal asymptotes makes sense; the graph is symmetric with respect to the *x*-axis. The equation of the relation remains unchanged when (x, y) is replaced by $(x, -y)$.

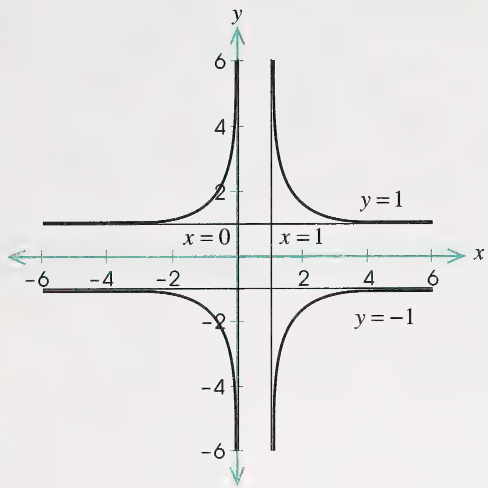
$(-y)^2\left(x^2-x\right)=x^2+4$ is the same as $y^2\left(x^2-x\right)=x^2+4$.

Since $y^2 = \frac{x^2+4}{x(x-1)}$, and y^2 is non-negative, $x(x-1)$ must be positive.

The function is only defined if $x > 1$ or $x < 0$.

The domain is $(-\infty, 0) \cup (1, \infty)$.

The graph is as follows:



Use Example 4 as a guide to answer question 2.

Use a graphing calculator to check your answer to question 2.

2. Sketch the relation $y^2 = \dfrac{x}{x-2}$.

Check your answer by turning to the Appendix.

Science fiction writers speculate that advanced technological societies may invent spacecraft that can exceed the speed of light. Would the function describing the energy required to propel those crafts have a vertical asymptote at the speed of light? Why not?

Activity 3: Oblique Asymptotes

In the preceding activities, you investigated methods for determining horizontal and vertical asymptotes. This activity deals with finding **oblique** or **slant asymptotes**—asymptotes which are neither horizontal nor vertical.

The functions you will study initially are rational functions.

Rational functions are functions of the form $y = \frac{P(x)}{Q(x)}$, where both $P(x)$ and $Q(x)$ are polynomials.

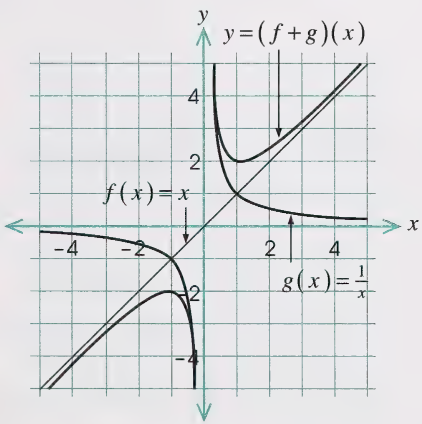
Example 1

Given $f(x)=x$, $g(x)=\frac{1}{x}$, determine $y=(f+g)(x)$. How are the graphs of the functions *f*, *g*, and $f+g$ related?

Solution

$$(f+g)(x)=f(x)+g(x)$$
$$=x+\frac{1}{x}$$
$$=\frac{x^2+1}{x}$$

Consider the graphs of all three functions.



Near the origin, the graphs appear quite different. However, as you move along either branch of $y=x+\frac{1}{x}$, as $x\to\pm\infty$, the graph approaches $y=x$. This makes sense, because $y=x+\frac{1}{x}$ is simply $y=x$ adjusted by adding $\frac{1}{x}$. As *x* approaches $\pm\infty$, then $\frac{1}{x}$ approaches 0, contributing less and less to the sum. This fact becomes apparent when you look at the following tables of values.

x	$\frac{1}{x}$	$x+\frac{1}{x}$
1	1	2
10	0.1	10.1
100	0.01	100.01
1000	0.001	1000.001

x	$\frac{1}{x}$	$x+\frac{1}{x}$
−1	−1	−2
−10	−0.1	−10.1
−100	−0.01	−100.01
−1000	−0.001	−1000.001

The line $y=x$, which the curve approaches, is the slant or oblique asymptote. How can you tell if a rational function has a slant asymptote, and how can you determine the equation of that slant asymptote from the equation of the curve?

Example 2

Determine the oblique asymptote of $y=\dfrac{x^2+1}{x+1}$. Sketch the graph.

Solution

As in Example 1, there will be an oblique asymptote if this rational function can be rewritten in the form $y=mx+b+(\text{fractional part})$, and if the fractional part approaches 0 as x becomes infinite. The slant asymptote will be $y=mx+b$.

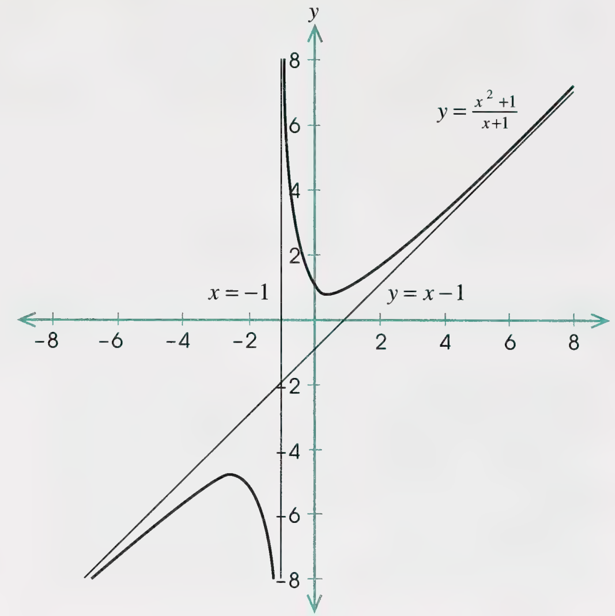
Use long division.

$$\begin{array}{r}
x-1 \\
x+1\overline{\smash{)}\,x^2+0x+1} \\
\underline{x^2+1x} \\
-1x+1 \\
\underline{-1x-1} \\
2
\end{array}$$

$$\therefore\ \frac{x^2+1}{x+1}=x-1+\frac{2}{x+1}$$

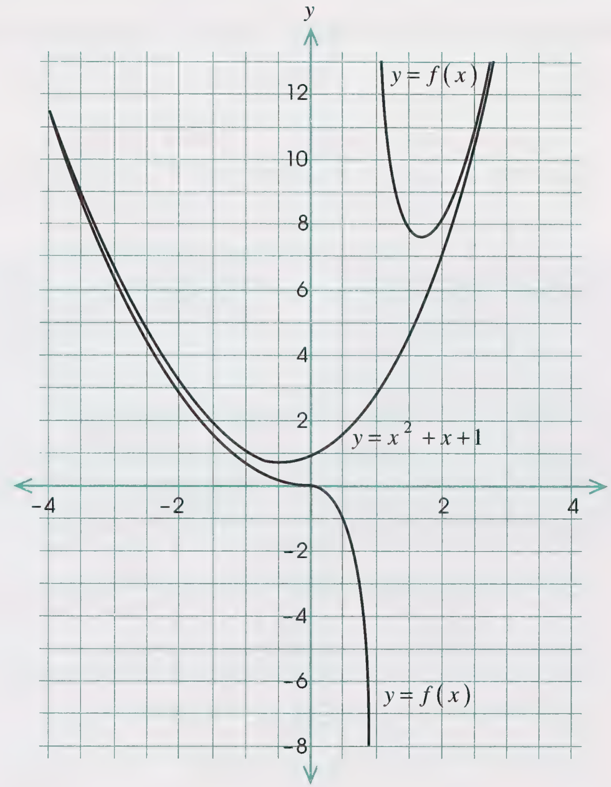
Since the function can now be expressed as $y=x-1+\frac{2}{x+1}$, the oblique asymptote must be $y=x-1$. The fractional part $\frac{2}{x+1}\to 0$ as $x\to\infty$.

Compare the graphs of $y=\frac{x^2+1}{x+1}$ and $y=x-1$.



Example 2 illustrates that for a rational function $y = \frac{P(x)}{Q(x)}$ to have an oblique asymptote, the degree of the numerator $P(x)$ must be one more than the degree of the denominator $Q(x)$. For instance, $y = f(x) = \frac{x^3}{x-1}$ would not have an oblique asymptote. The long division method shows this.

$$
\begin{array}{r}
x^2 + x + 1 \\
x-1 \overline{) x^3 + 0x^2 + 0x + 0} \\
\underline{x^3 - 1x^2} \\
1x^2 + 0x \\
\underline{1x^2 - 1x} \\
1x + 0 \\
\underline{1x - 1} \\
1
\end{array}
$$



The function when rewritten is $y = x^2 + x + 1 + \frac{1}{x-1}$. Clearly, as $x \to \infty$, the function approaches $y = x^2 + x + 1$ which is not a line. However, the graph of the function does approach the parabola $y = x^2 + x + 1$, and the function is said to be asymptotic to that curve. In some instances, you may wish to use this as an aid in graphing the original function.

1. For each of the following, determine the vertical, horizontal, and oblique asymptotes, if any. Use the asymptotes found, and a graphing calculator, if necessary, to sketch each curve.

   a. $y = 3x - 2 + \dfrac{1}{x-2}$
   b. $y = \dfrac{x^2 + x}{x - 2}$
   c. $y = \dfrac{x^3}{x^2 - 4}$

2. Can a rational function have both slant (oblique) and horizontal asymptotes? Explain.

Check your answers by turning to the Appendix.

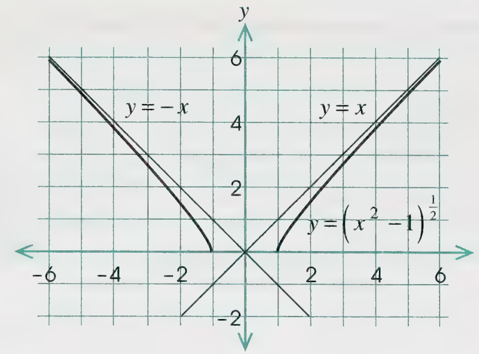
Besides rational functions, an example of other functions that may have oblique asymptotes is a function of the form $y = \left(ax^2 + b\right)^{\frac{1}{2}}$. Consider the following example.

Example 3

Find the oblique asymptotes of $y = \left(x^2 - 1\right)^{\frac{1}{2}}$.

Solution

Compare the graphs of $y = \left(x^2 - 1\right)^{\frac{1}{2}}$ and $y = \pm x$.



Why do the branches of this function approach $y = \pm x$? Consider the following tables of values.

x	y
10	9.949 87
100	99.995 00
1000	999.999 50

x	y
−10	9.949 87
−100	99.995 00
−1000	999.999 50

From the tables, it is apparent that as $x \to +\infty$, $\left(x^2 - 1\right)^{\frac{1}{2}} \to x$; as $x \to -\infty$, $\left(x^2 - 1\right)^{\frac{1}{2}} \to -x$.

The asymptotes are $y = x$ and $y = -x$.

Example 3 illustrates that $\left(x^2 \pm c\right)^{\frac{1}{2}} \to x$ as $x \to +\infty$, and $\left(x^2 \pm c\right)^{\frac{1}{2}} \to -x$ as $x \to -\infty$.

Example 4

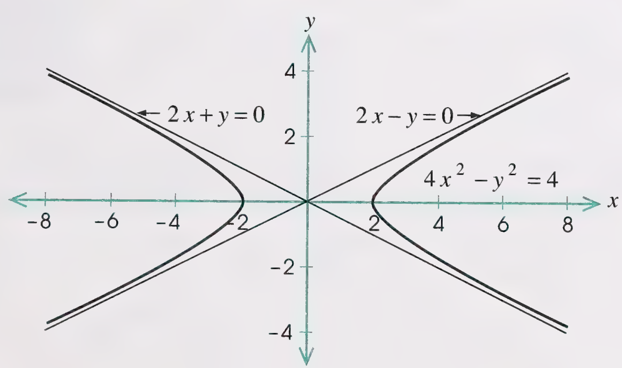
Find the asymptotes of $x^2 - 4y^2 = 4$; then, sketch the graph.

Solution

Solve for y.

$$x^2 - 4y^2 = 4$$
$$4y^2 = x^2 - 4$$
$$2y = \pm\left(x^2 - 4\right)^{\frac{1}{2}}$$

As x increases or decreases without bound, the curve approaches $2y = \pm x$ or $y = \pm\frac{1}{2}x$ (the oblique asymptotes).



3. Find the slant asymptotes of each relation; then, sketch the graphs.

a. $y = 3\left(x^2 + 4\right)^{\frac{1}{2}}$ **b.** $y^2 - 9x^2 = 4$

Check your answers by turning to the Appendix.

Remember, a rational function cannot have both horizontal and oblique asymptotes. There are, of course, other functions and relations that have both. Can you think of such a function?

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You may find it helpful to view the video titled *Derivatives and Graph Sketching* from the *Catch 31* series, ACCESS Network. This segment reviews the procedures for finding vertical and horizontal asymptotes. It also previews what you will study in the later sections. This video is available from the Learning Resources Distributing Centre.

When working with rational functions, you used different techniques for finding vertical asymptotes than for finding horizontal asymptotes. For a rational function of the form $y = \frac{P(x)}{Q(x)}$, you found the vertical asymptotes by determining the zeros of $Q(x)$. To find the horizontal asymptotes, you determined $\lim\limits_{x\to\infty} \frac{P(x)}{Q(x)}$.

If that function could be rewritten as $x = \frac{N(y)}{D(y)}$, then it should be possible to switch procedures. To find the horizontal asymptotes, determine the zeros of $D(y)$. To find the vertical asymptotes, determine $\lim\limits_{y\to\infty} \frac{N(y)}{D(y)}$.

Example

Find the vertical and horizontal asymptotes of $y = \frac{3x-2}{x-4}$. Sketch the graph.

Solution

First, find the horizontal asymptotes.

Method 1: Using Limits

$$\begin{aligned}\lim_{x\to\infty} \frac{3x-2}{x-4} &= \lim_{x\to\infty} \frac{x\left(3-\frac{2}{x}\right)}{x\left(1-\frac{4}{x}\right)}\\ &= \frac{3-0}{1-0}\\ &= 3\end{aligned}$$

The horizontal asymptote is $y = 3$.

Method 2: Finding the Zeros of the Denominator

Solve the equation for x and find the zeros of the denominator of the resulting fractional expression.

$$\begin{aligned}y &= \frac{3x-2}{x-4}\\ y(x-4) &= 3x-2\\ xy-4y &= 3x-2\\ xy-3x &= 4y-2\\ x(y-3) &= 4y-2\\ x &= \frac{4y-2}{y-3}\end{aligned}$$

Since the denominator of the result is zero when $y = 3$, then the line $y = 3$ must be the horizontal asymptote.

Next, find the vertical asymptote.

Method 1: Finding the Zeros of the Denominator

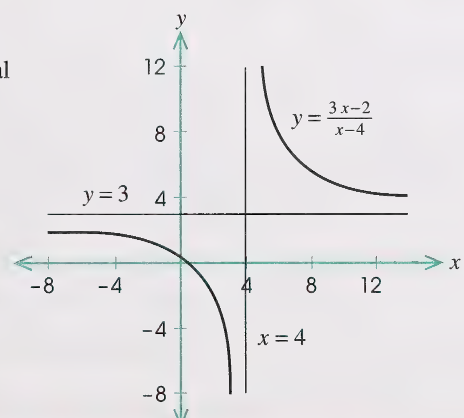
For $y = \frac{3x-2}{x-4}$, the denominator is zero when $x = 4$; therefore, $x = 4$ is the vertical asymptote.

Method 2: Using Limits

Find the limit as $y \to \infty$ of $x = \frac{4y-2}{y-3}$.

$$\begin{aligned}\lim_{y\to\infty} \frac{4y-2}{y-3} &= \lim_{y\to\infty} \frac{y\left(4-\frac{2}{y}\right)}{y\left(1-\frac{3}{y}\right)} \\ &= \frac{4-0}{1-0} \\ &= 4\end{aligned}$$

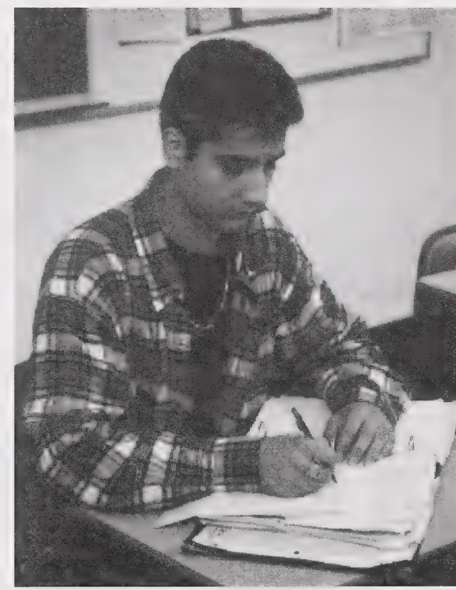
Therefore, the vertical asymptote is $x = 4$.



1. Find the asymptotes of $y = \frac{x}{1-x}$ using both methods. Then sketch the graph.

2. Find the asymptotes of $y = \frac{1-2x}{x-2}$ using both methods. Then sketch the graph.

Check your answers by turning to the Appendix.



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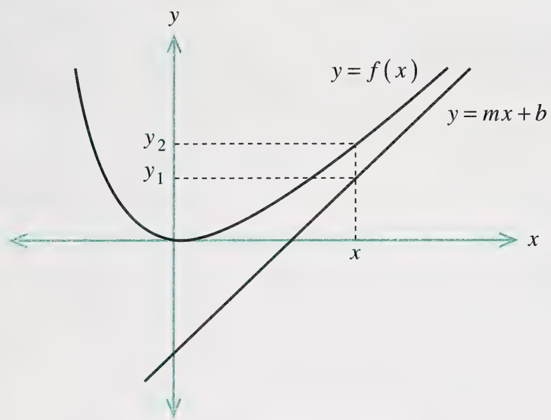
Enrichment

If a function $y = f(x)$ has an oblique (or slant) asymptote $y = mx + b$, then the vertical distance between the graph of the function and the oblique line must approach 0 as $x \to \infty$.



In the diagram, $\lim_{x\to\infty}(y_2 - y_1) = 0$.

This provides a method for verifying whether or not a particular line $y = mx + b$ is an oblique asymptote of $y = f(x)$. You must show $\lim_{x\to\infty}\left[f(x) - (mx + b)\right] = 0$.

Example 1

Show that $y = x - 4$ is an oblique asymptote of $f(x) = \frac{x^2}{x+4}$.

Solution

You must show $\lim_{x\to\infty}\left[f(x) - (x-4)\right]$ is equal to 0.

$$\begin{aligned}
\lim_{x\to\infty}\left[f(x)-(x-4)\right] &= \lim_{x\to\infty}\left[\frac{x^2}{x+4}-(x-4)\right] \\
&= \lim_{x\to\infty}\left[\frac{x^2-(x-4)(x+4)}{x+4}\right] \\
&= \lim_{x\to\infty}\left[\frac{x^2-x^2+16}{x+4}\right] \\
&= \lim_{x\to\infty}\frac{16}{x+4} \\
&= \lim_{x\to\infty}\frac{x\left(\frac{16}{x}\right)}{x\left(1+\frac{4}{x}\right)} \\
&= \frac{0}{1+0} \\
&= 0
\end{aligned}$$

Therefore, $y = x - 4$ is an oblique symptote.

This technique can be used to verify the oblique asymptotes of a hyperbola.

Example 2

Verify that $y = +\frac{a}{b}x$ is a slant asymptote of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Assume the parameters a and b are positive.

Solution

Begin by solving the equation of the hyperbola for y in terms of x.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{a^2} = \frac{x^2}{b^2} + 1$$

$$\frac{y^2}{a^2} = \frac{x^2 + b^2}{b^2}$$

$$y^2 = \frac{a^2}{b^2}\left(x^2 + b^2\right)$$

$$y = \pm\frac{a}{b}\left(x^2 + b^2\right)^{\frac{1}{2}}$$

You must now decide whether to use the positive or the negative expression with $y = +\frac{a}{b}x$. If you are investigating what happens when $x \to +\infty$, assume $x > 0$. If $x > 0$, then $\frac{a}{b}x > 0$. Therefore, select the positive expression.

You must now show $\lim\limits_{x\to\infty}\left[\frac{a}{b}x - \frac{a}{b}\sqrt{x^2 - b^2}\right] = 0$.

$$\begin{aligned}
&\lim_{x\to\infty}\left[\frac{a}{b}x - \frac{a}{b}\sqrt{x^2 - b^2}\right] \\
&= \lim_{x\to\infty}\frac{a}{b}\left[x - \sqrt{x^2 - b^2}\right] \\
&= \frac{a}{b}\lim_{x\to\infty}\left[x - \sqrt{x^2 - b^2}\right] \\
&= \frac{a}{b}\lim_{x\to\infty}\frac{\left[x - \sqrt{x^2 - b^2}\right]\left[x + \sqrt{x^2 - b^2}\right]}{\left[x + \sqrt{x^2 - b^2}\right]} \\
&= \frac{a}{b}\lim_{x\to\infty}\frac{\left[x^2 - \left(x^2 - b^2\right)\right]}{x + \sqrt{x^2 - b^2}} \\
&= \frac{a}{b}\lim_{x\to\infty}\frac{b^2}{\left[x + \sqrt{x^2 - b^2}\right]} \\
&= ab\lim_{x\to\infty}\frac{1}{\left[x + \sqrt{x^2 - b^2}\right]} \\
&= ab(0) \qquad \text{(since the denominator becomes infinite)} \\
&= 0
\end{aligned}$$

Similarly, $y = -\frac{a}{b}x$ can be verified as a slant asymptote.

1. Verify that $y = x$ is an oblique asymptote of $y = \frac{x^3+1}{x^2}$.

2. Show that $y = x - 1$ is an asymptote of $y = \frac{x^3-x^2-1}{x^2+1}$.

Check your answers by turning to the Appendix.

Conclusion

You should now be able to define and discuss the conditions under which asymptotes occur. Also, you should now be able to determine vertical asymptotes of algebraic functions from the zeros of their denominators, find horizontal asymptotes using limits as $x \to \pm\infty$, and use long division to rewrite functions to locate oblique or slant asymptotes.

In this section, you discovered that asymptotes are like boundary lines a curve approaches, as either the independent or dependent variable becomes infinite. Even formulas from the physical sciences may involve asympotes. Relativity theory states that the mass of a particle travelling at a speed exceeding approximately one-tenth of the speed of light is significantly larger than its rest mass. The mass of the particle becomes infinite as it approaches the speed of light. If this theory is correct, travel to the stars becomes practically impossible since the speeds spacecraft can travel is limited, but the distances are extreme.

You may wish to review how asymptotes assist you in sketching the graph of a particular function or relation.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 3: The First Derivative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When driving through the mountains, sometimes the appearance of the terrain makes it difficult to decide whether a particular stretch of highway is rising, falling, or perfectly level. So, too, is it difficult to determine whether a particular part of a function is increasing, decreasing, or stationary. When sketching the curve of a function, you need to determine precisely where that function increases, decreases, or remains stationary. In addition, you will need to know where extreme values occur—where the curve has maximum points and where the curve has minimum points.

In this section you will use the First Derivative Test to help locate maximum and minimum values. You will decide whether a local minimum or maximum is an overall minimum or maximum, and you will be given guidelines as to where to look for extreme values.

You will use the sign of the first derivative to decide where a particular curve is increasing or decreasing, and how that information can be used to sketch the curve.

In the last activity you will find out what you can or cannot assume about the graph of a function from its first derivative.

As you will see, calculus is a powerful tool in the determination of maximum and minimum points and in the sketching of curves.

Activity 1: Intervals of Increase or Decrease

You are a theatre manager responsible for setting ticket prices for an upcoming production. From past experience, you know if you charge \$8.00 per ticket, 800 people will attend. You also know that if you raise the ticket price, fewer people will attend. If 50 fewer people would attend for every \$1.00 increase in ticket price, what should you charge to maximize your financial return?

Set up a table to analyse the problem.

Price per Ticket	Number of Tickets	Financial Return
\$8.00	800	$\$8(800) = \$6400.00$
\$9.00	750	$\$9(750) = \$6750.00$
\$10.00	700	\$7000.00
\$11.00	650	\$7150.00
\$12.00	600	\$7200.00
\$13.00	550	\$7150.00
\$14.00	500	\$7000.00
\$15.00	450	\$6750.00
\$16.00	400	\$6400.00

Initially, as you raise the ticket price, your return increases. You appear to make the most money at \$12.00 per ticket. Further increases in ticket price decrease your return.

In this and in other situations, you are expected to find intervals where the functions involved increase or decrease and where (or if) extreme values occur.

What function can be used to model the preceding situation?

Let y represent the financial return; and let x represent the ticket price.

Initially the number of tickets sold is 800. However, the number sold declines by 50 for every dollar the price exceeds $8.00.

Now, $(x-8)$ is the price increase.

$50(x-8)$ is the number of fewer people in attendence.

Therefore, $800-50(x-8)$ is the number of tickets sold.

$$\begin{aligned}\text{financial return} &= (\text{ticket price})\times(\text{number of tickets})\\ y &= x[800-50(x-8)]\\ &= x[800-50x+400]\\ &= x(-50x+1200)\\ &= -50x^2+1200x\end{aligned}$$

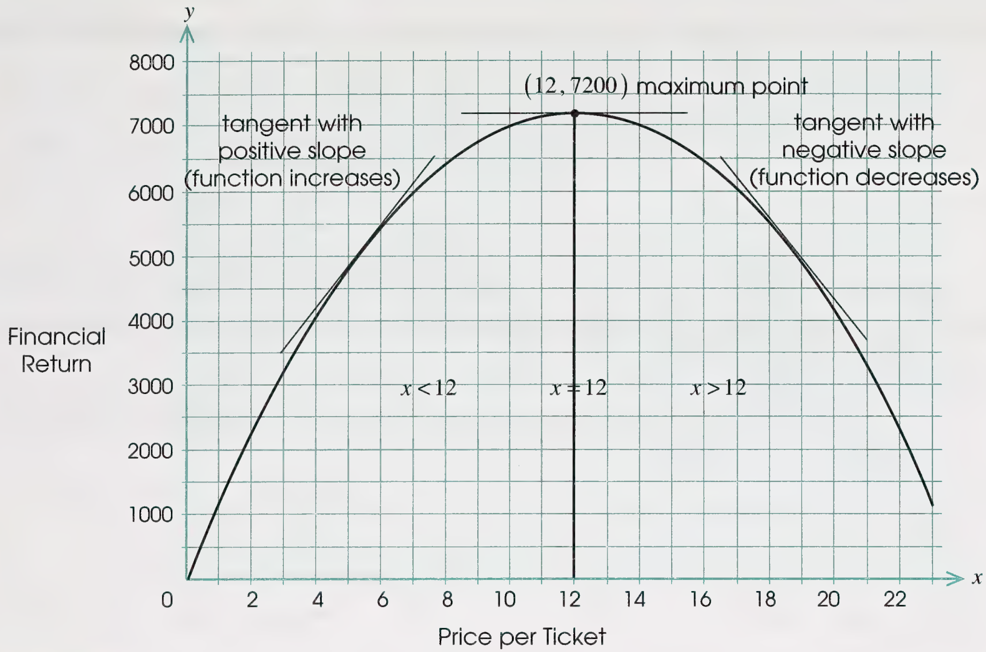
In Example 1, you will study this function further. For ease of analysis, you will assume this is a **continuous function** defined on the set of reals.

Example 1

Determine the intervals where the function $y=-50x^2+1200x$ increases, decreases, or is a maximum.

Solution

Method 1: Analysing the Graph



From the graph, it appears that y increases when $x < 12$, decreases when $x > 12$, and is a maximum when $x = 12$. If tangent lines were drawn to the curve, at points where the graph increases, the slope of the tangent line would be positive; at points where the graph decreases, the slope of the tangent would be negative; at the maximum, the slope would be zero.

The terms increasing function and decreasing function refer to what happens to y as x increases. The function is said to increase if y gets larger as x gets larger; the function is said to decrease if y gets smaller as x gets larger. Often the term ***rising*** is used to describe the graph of an increasing function, and the term ***falling*** is used to describe the graph of a decreasing function.

Method 2: Using the Derivative

The derivative represents the slope of a tangent line. If the slope of a tangent is positive, the function is increasing at this point; therefore, the function increases if $\frac{dy}{dx} > 0$.

$$y = -50x^2 + 1200x$$

$$\begin{aligned}\frac{dy}{dx} &= -50(2x) + 1200(1)\\ &= -100x + 1200\end{aligned}$$

If $\frac{dy}{dx} > 0$, then
$$\begin{aligned}-100x + 1200 &> 0\\ 1200 &> 100x\\ 12 &> x\\ x &< 12\end{aligned}$$

Therefore, the function increases when $x < 12$, or in the interval $(-\infty, 12)$.

If the slope of a tangent is negative, the function is decreasing at this point. Therefore, the function decreases if $\frac{dy}{dx} < 0$.

If $\frac{dy}{dx} < 0$, then
$$\begin{aligned}-100x + 1200 &< 0\\ 1200 &< 100x\\ 12 &< x\\ x &> 12\end{aligned}$$

Therefore, the function decreases when $x > 12$, or in the interval $(12, \infty)$.

In this example, the maximum point occurs when $\frac{dy}{dx} = 0$. That is at the transition point between an increasing function (positive slope) and a decreasing function (decreasing slope). When $\frac{dy}{dx} = 0$, then $-100x + 1200 = 0$ or $x = 12$.

Now,
$$\begin{aligned}f(12) &= -50(12)^2 + 1200(12)\\ &= -7200 + 14\,400\\ &= 7200\end{aligned}$$

Therefore, 7200 is the maximum of the function.

Knowing where functions increase, decrease, and have maximum or minimum values is vital when sketching the graphs of those functions.

Example 2

Determine the intervals where the function $f(x) = x + \frac{1}{x}$ increases, decreases, and has maximum or minimum points. Use this information, together with the domain, range, and asymptotes (if any), to sketch the curve.

Solution

Determine the intervals where the function increases.

$$f(x) = x + \frac{1}{x} \text{ or } f(x) = x + x^{-1}$$

$$f'(x) = 1 - 1x^{-2}$$

When $f'(x) > 0$, $1 - 1x^{-2} > 0$

$$\begin{aligned} 1 &> x^{-2} \\ 1 &> \frac{1}{x^2} \\ x^2 &> 1 \end{aligned}$$

Therefore, the function increases when $x > 1$ or $x < -1$; that is, it increases in the interval $(-\infty, -1) \cup (1, \infty)$.

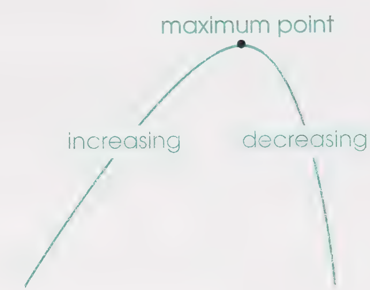
Determine the intervals where the function decreases.

When $f'(x) < 0$, $1 - 1x^{-2} < 0$

$$\begin{aligned} 1 &< x^{-2} \\ 1 &< \frac{1}{x^2} \\ x^2 &< 1 \end{aligned}$$

Now, $x \neq 0$ because the original function and the derivative would be undefined. Therefore, the function decreases when $-1 < x < 0$ or $0 < x < 1$; that is, it decreases in the interval $(-1, 0) \cup (0, 1)$.

At $x = -1$, the function changes from an increasing to a decreasing function; a maximum point occurs at $x = -1$.

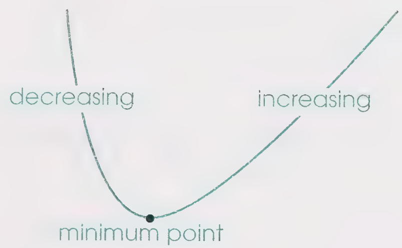


When $x = -1$, $y = -1 + \frac{1}{-1}$

$$= -2$$

Therefore, $(-1, -2)$ is a maximum point.

At $x = 1$, the function changes from a decreasing to an increasing function; thus, a minimum point occurs at $x = 1$.



When $x = 1$, $y = 1 + \frac{1}{1}$
$= 2$

Therefore, $(1, 2)$ is a minimum point.

Notice that the tangent is horizontal at both the maximum and minimum points.

When $\frac{dy}{dx} = 0$, $1 - 1x^{-2} = 0$

$$1 = \frac{1}{x^2}$$

$$x^2 = 1$$

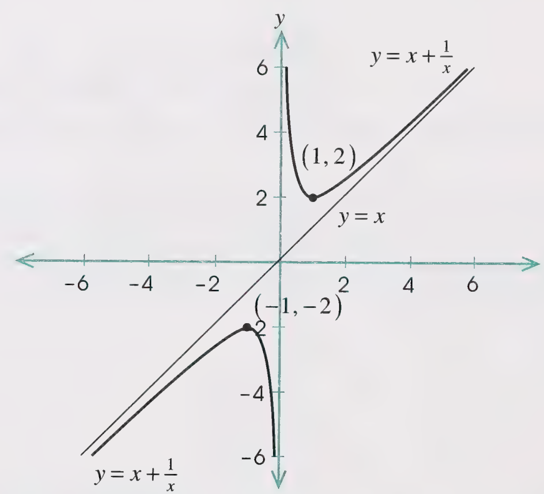
$$x = \pm 1$$

Now, the domain of the original function is $\{x \mid x \neq 0\}$. Therefore, $x < 0$ or $x > 0$.

When $x < 0$, the function increases to point $(-1, -2)$; it then decreases. The range when $x < 0$ is $y < -2$. When $x > 0$, the function decreases to $(1, 2)$ and then increases; the range is $y > 2$.

Therefore, the range is $(-\infty, -2) \cup (2, \infty)$.

Since the function $y = x + \frac{1}{x}$ is undefined when $x = 0$, a vertical asymptote occurs when $x = 0$. As x increases or decreases without bound, the function approaches the line $y = x$. This line is an oblique asymptote. The graph is as follows:



To summarize, given $y = f(x)$, the function increases when $\frac{dy}{dx} > 0$; that is, its graph rises in that interval. The function decreases when $\frac{dy}{dx} < 0$; that is, its graph falls in that interval. For a **continuous function**, at points of transition in the graph, where the slope is changing from positive to negative or negative to positive, maximum or minimum values occur. The slope at these points may be zero. This is known as the **First Derivative Test**.

1. Determine the values of x for which the curve is rising or falling. Also, determine any maximum or minimum points. Sketch each curve. Use intercepts and asymptotes (if any) to assist you.

a. $y = x^2 + 2x - 3$ **b.** $y = -2x^2 + 4x - 2$

c. $y = x + \frac{9}{x}$ **d.** $y = \frac{x}{x+1}$

Check your answers by turning to the Appendix.

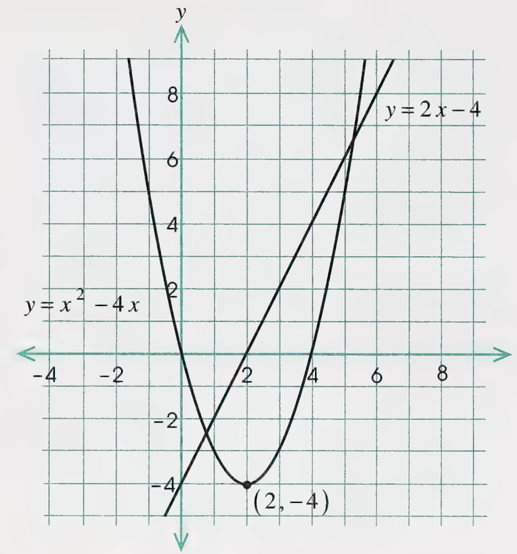
In Example 1 and Example 2, the extreme values occur at points where the slope changes from positive to negative, or vice versa. If the derivative is defined at these points, then $\frac{dy}{dx} = 0$. Later, you will see that this is not the only condition for which maximum or minimum values occur. Also, just because the derivative is zero does not mean the curve necessarily has a maximum or minimum. However, you now have an aid in determining extreme values.

Example 3

Compare the graphs of $f(x) = x^2 - 4x$ and its derivative.

Solution

$f(x) = x^2 - 4x$
$f'(x) = 2x - 4$



The graph of $f(x)=x^2-4x$ falls when $x<2$, rises when $x>2$, and has a minimum point at $x=2$.

The graph of $f'(x)=2x-4$ lies below the *x*-axis when $x<2$. This means the slopes of the tangents to *f* in this interval are negative (*f* decreases).

To check , solve $f'(x)<0$.

$$\begin{aligned} 2x-4&<0 \\ 2x&<4 \\ x&<2 \end{aligned}$$

Now, the graph of $f'(x)=2x-4$ lies above the *x*-axis when $x>2$. This means the slopes of the tangents to *f* in this interval are positive (*f* increases).

To check, solve $f'(x)>0$.

$$\begin{aligned} 2x-4&>0 \\ 2x&>4 \\ x&>2 \end{aligned}$$

The graph of $f'(x)=2x-4$ intersects the *x*-axis at $x=2$; that is, $f'(2)=0$.

The slope of the tangent to $f(x)=x^2-4x$ at $x=2$ is horizontal. The function has a minimum point at $x=2$.

$$\begin{aligned} f(2)&=2^2-4(2) \\ &=-4 \end{aligned}$$

Therefore, the minimum point is $(2,-4)$.

Example 4

Determine the values of *x* for which $f(x)=x^3+2x^2-4x-8$ increases or decreases. What are the maximum and minimum points? Sketch both the original function and its derivative. Compare the graphs.

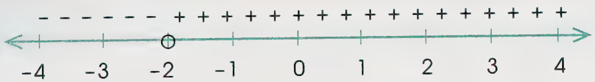
Solution

$$f(x)=x^3+2x^2-4x-8$$
$$f'(x)=3x^2+4x-4$$

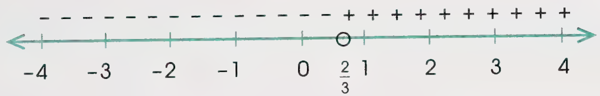
Function *f* increases when $f'(x)>0$.

$$\begin{aligned} 3x^2+4x-4&>0 \\ (3x-2)(x+2)&>0 \end{aligned}$$

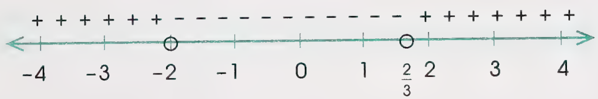
Sign of $(x+2)$



Sign of $(3x-2)$



Sign of $(3x-2)(x+2)$



The function f is increasing when $x<-2$ or $x>\frac{2}{3}$; thus, the graph rises for values of x in the interval $\left(-\infty,-2\right)\cup\left(\frac{2}{3},\infty\right)$.

Function f decreases when $f'(x)<0$.

$$3x^2+4x-4<0$$
$$(3x-2)(x+2)<0$$

The function f is decreasing when $-2<x<\frac{2}{3}$; thus, the graph falls for values of x in the interval $\left(-2,\frac{2}{3}\right)$.

For this function, maximum or minimum values should occur when $f'(x)=0$.

$$3x^2+4x-4=0$$
$$(3x-2)(x+2)=0$$

$$3x-2=0 \quad \text{or} \quad x+2=0$$
$$x=\frac{2}{3} \qquad\qquad x=-2$$

$$f\left(\frac{2}{3}\right)=\left(\frac{2}{3}\right)^3+2\left(\frac{2}{3}\right)^2-4\left(\frac{2}{3}\right)-8$$
$$=\frac{8}{27}+\frac{8}{9}-\frac{8}{3}-8$$
$$=-\frac{256}{27}$$

To the immediate left of $x=\frac{2}{3}$, the function is falling; to the immediate right of $x=\frac{2}{3}$, the function is rising. Therefore, $\left(\frac{2}{3},-\frac{256}{27}\right)$ is a minimum point.

$$
\begin{aligned}
f(-2) &= (-2)^3 + 2(-2)^2 - 4(-2) - 8 \\
&= -8 + 8 + 8 - 8 \\
&= 0
\end{aligned}
$$

To the immediate left of $x = -2$, the function is rising; to the immediate right of $x = -2$, the function is falling. Therefore, $(-2, 0)$ is a maximum point.

The intercepts are often useful in sketching the curve. Clearly, the graph of $y = f(x)$ is tangent to the x-axis at $x = -2$. This value may be used to determine the other intercept. Divide $x^3 + 2x^2 - 4x - 8$ by $x + 2$ synthetically.

$$
\begin{array}{r|rrrr}
-2 & 1 & 2 & -4 & -8 \\
 & & -2 & 0 & 8 \\
\hline
 & 1 & 0 & -4 & \boxed{0}
\end{array}
$$

$$
\begin{aligned}
\therefore x^3 + 2x^2 - 4x - 8 &= (x+2)\left(x^2 - 4\right) \\
&= (x+2)(x+2)(x-2) \\
&= (x+2)^2(x-2)
\end{aligned}
$$

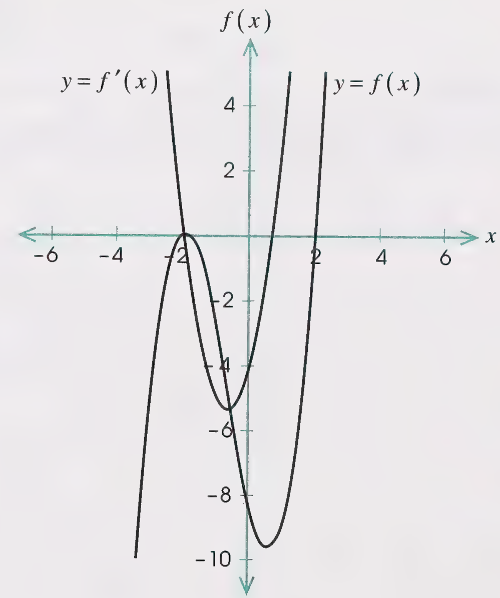
The graph touches the x-axis at $(-2, 0)$ and crosses the x-axis at $(2, 0)$.

To find the y-intercept, let $x = 0$.

$$
\begin{aligned}
y &= 0^3 + 2(0)^2 - 4(0) - 8 \\
&= -8
\end{aligned}
$$

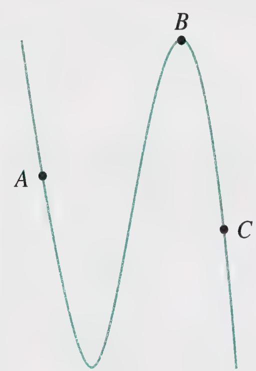
The graph crosses the y-axis at $(0, -8)$.

Now sketch the curves.



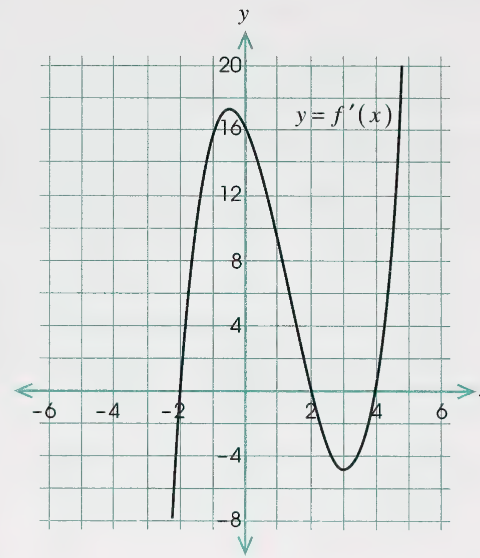
When you compare the graphs of $y = f(x)$ and $y = f'(x)$, you will notice that the portion of the graph of $y = f'(x)$ that lies above the x-axis corresponds to that portion of $y = f(x)$ which rises to the right; when $f'(x) > 0$, $f(x)$ increases. You will also notice that the portion of the graph of $y = f'(x)$ that lies below the x-axis corresponds to that portion of $y = f(x)$ which falls to the right; when $f'(x) < 0$, $f(x)$ decreases. The x-intercepts of the graph of $y = f'(x)$, indicating where $f'(x) = 0$, correspond to the maximum and minimum points of $y = f(x)$.

2. For points A, B, and C on the graph of $y = f(x)$, state whether the function is increasing, decreasing, or has a maximum or minimum. Also state whether the slope of the curve at those points is positive, negative, or zero.



3. For the function $f(x) = -x^3 + 12x - 1$, state where f increases, decreases, or has a maximum or minimum value. Then sketch the graphs of $f(x)$ and $f'(x)$.

4. The following is the graph of the derivative of a certain function. Indicate the intervals where the original function is increasing, decreasing, or has maximum or minimum values.



Check your answers by turning to the Appendix.

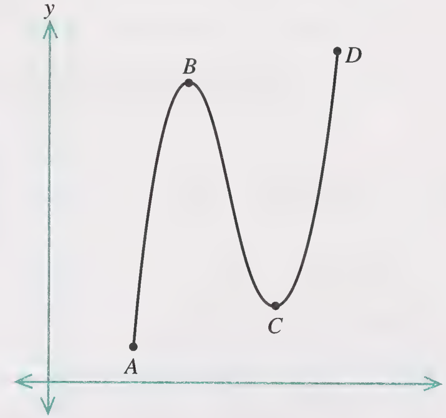
Remember, the fact that the slope is positive or negative only tells you if the curve rises or falls. Precisely how steep the curve is at a particular point is determined from the magnitude of the slope. Think of a mountain road rising or falling. The gradient (slope) will indicate exactly how much the road rises or falls.

Activity 2: Extreme Values

Have you ever reached for an object underwater, only to find your fingers close some distance from the object? Have you done a chemistry experiment where a metal strip appears to be bent in a glass of water?

What does this problem have in common with fencing a pasture, manufacturing an oil tin, finding the distance a projectile will travel, determining business profits, and calculating the strength of a beam or the efficiency of machines? All of these problems deal with maximum and minimum values—the maximum area enclosed by a given length of fencing, the least amount of material necessary to construct an oil tin of a given volume, the maximum distance a projectile will travel, the highest possible profits, the strongest beam, the most efficient machine, and, in the case of the object in the water or the metal strip in the glass, the least time required for light to travel through both air and water.

Being able to determine **extreme** values (maxima and minima) is an essential skill in calculus. However, before looking at a few examples, you will need to know some terminology.



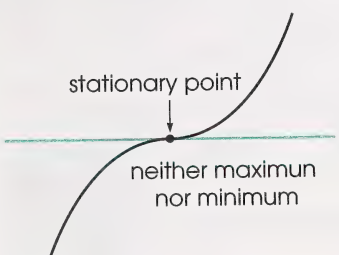
In the preceding diagram, A, B, C, and D are all **extrema**: maximum and minimum points at which the function assumes maximum or minimum values. The value of y is a minimum at A, a maximum at B, and so on.

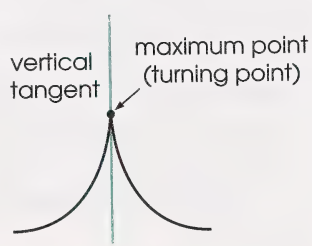
However, there is only one **absolute maximum** in the diagram—the value of y at D; all other values of the function are less. Similarly, there is only one **absolute minimum**—$f(x)$ at A; all other values of the function are larger. As you can see from the diagram, it will be necessary to check values at the endpoints of the interval, on which the function is defined, for extreme values.

Points A and C are known as a **relative (local) minima**, and points B and D are known as a **relative (local) maxima**. This means that in the immediate neighbourhood of each point, those values of the function, when compared relative to others in those localities, are minimum or maximum values. Point C, for example, is the lowest point on the graph, when compared to others close by.

As you saw in the previous activity, maximum and minimum values often occur at transition points where the slopes change sign. As a result, these points are often called **turning points**. When a continuous function ceases to increase and begins to decrease (as at point B), it is said to have a maximum value. When a function ceases to decrease and begins to increase (as at point C), it is said to have a minimum value. If the slope at those points is 0 (as it appears to be at both B and C), the slope is neither increasing nor decreasing. In that case, those points are referred to as **stationary points**.

Not all stationary points are maximums or minimums, and not all maximum or minimum points have horizontal tangents—the slope at some maximum or minimum points may even be infinite. (You will study this later.)



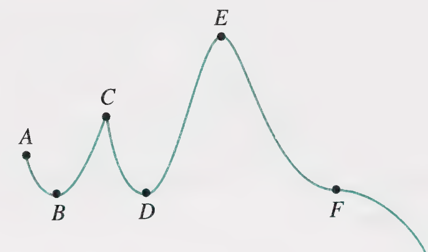


Nevertheless, as part of the process of identifying extreme values, you will be looking for values of x to locate points for which the function has horizontal (or vertical tangents. These values of x will be called **critical values**.

Practise using the terminology.

1. Match each term with the appropriate point(s) from the diagram.

- relative minimum
- relative maximum
- absolute minimum
- absolute maximum
- turning point
- stationary point



2. The derivative of a function $y = f(x)$ is $f'(x) = x^2 - 3x + 2$. What are the critical values of x?

3. Determine the critical values of x for $y = x^{\frac{1}{3}}$.

Check your answers by turning to the Appendix.

Now that you are familiar with the terminology, you will use those terms in analysing functions.

Example 1

Find all maximum or minimum values of $f(x) = x^2 - 3x + 2$, where $x \geq -1$.

Solution

Begin by checking for stationary points, since extreme values may occur when $f'(0) = 0$.

Now, $f'(x) = 2x - 3$

If $f'(x) = 0$, then $2x - 3 = 0$

$$x = \frac{3}{2} \text{ or } 1.5$$

Find the value of the function corresponding to this critical value of x.

$$\begin{aligned} f\left(\frac{3}{2}\right) &= \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2 \\ &= \frac{9}{4} - \frac{9}{2} + 2 \\ &= -\frac{1}{4} \text{ or } -0.25 \end{aligned}$$

The point $\left(\frac{3}{2}, -\frac{1}{4}\right)$ might be a maximum point, a minimum point, or neither. How do you find out?

Method 1

Check values of the function in the neighbourhood, at points on **both** sides of $(1.5, -0.25)$. Choose $x = 1.4$ and $x = 1.6$. Determine the y-value corresponding to these x-values and compare the y-values with $y = -0.25$.

Remember, minima and maxima are y-values.

$$\begin{aligned} f(1.4) &= (1.4)^2 - 3(1.4) + 2 \\ &= -0.24, \text{ which is larger than } -0.25 \end{aligned}$$

$$\begin{aligned} f(1.6) &= (1.6)^2 - 3(1.6) + 2 \\ &= -0.24, \text{ which again is larger than } -0.25 \end{aligned}$$

Therefore, $y = -0.25$ is a relative minimum.

Method 2

Check on either side of the stationary point to see if the function is increasing or decreasing. The x-value of the stationary point is 1.5. Find $f'(1.4)$ and $f'(1.6)$.

$$\begin{aligned} f'(1.4) &= 2(1.4) - 3 \\ &= 2.8 - 3 \\ &= -0.2 \\ &< 0 \end{aligned}$$

The graph falls on the left.

$$\begin{aligned} f'(1.6) &= 2(1.6) - 3 \\ &= 3.2 - 3 \\ &= 0.2 \\ &> 0 \end{aligned}$$

The graph rises on the right.

Since the graph changes from a decreasing function to an increasing function, $y = -0.25$ is a minimum.

Since there are no other stationary points, the next step is to check the endpoint of the interval $[-1, \infty)$ for which the function is defined.

$$\begin{aligned} f(-1) &= (-1)2 - 3(-1) + 2 \\ &= 1 + 3 + 2 \\ &= 6 \end{aligned}$$

Is $(-1, 6)$ a maximum or minimum point?

Method 1

Check points in the neighbourhood. Since x cannot be less than -1, it is sufficient to check a larger value.

$$\begin{aligned} f(-0.9) &= (-0.9)^2 - 3(-0.9) + 2 \\ &= 0.81 + 2.7 + 2 \\ &= 5.51 \quad \text{(which is less than 6)} \end{aligned}$$

Therefore, $y = 6$ is a relative maximum.

Method 2

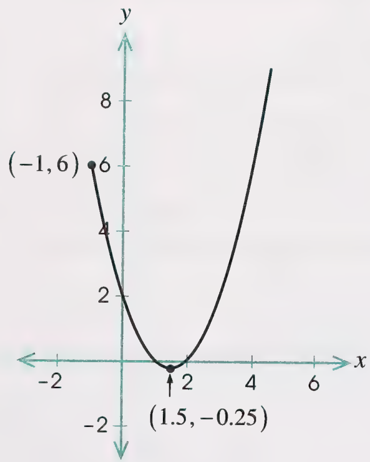
Check the slope to decide whether $f(-1) = 6$ is a maximum or minimum.

$$f'(-0.9) = 2(-0.9) - 3 < 0$$

Therefore, the graph falls on the right of $(-1, 6)$.

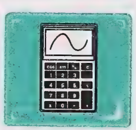
Therefore, $f(-1, 6)$ is a relative maximum.

Now, sketch the graph.



From the graph, you see that $f(1.5) = -0.25$ is an absolute minimum. There is no absolute maximum.

Example 2



Find the maximum and minimum values of the function of $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 - x$. Sketch the graph. Use a graphing calculator (or computer program) to check your answer.

Solution

As in Example 1, check for stationary points, if any. Remember, at a stationary point, $f'(x) = 0$.

$$f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 - x$$

$$f'(x) = \frac{1}{4}(4x^3) + \frac{1}{3}(3x^2) - \frac{1}{2}(2x) - 1$$

$$= x^3 + x^2 - x - 1$$

If $f'(x) = 0$, then $x^3 + x^2 - x - 1 = 0$.

List the factors of the constant.

The potential integral roots are ± 1.

Test $x = 1$ using synthetic division.

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -1 & -1 \\ & & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$$(x-1)(x^2 + 2x + 1) = 0$$

$$(x-1)(x+1)(x+1) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 1 \qquad x = -1$$

Therefore, the critical values of x are ± 1.

Evaluate the function for these values of x, and check to see if they are maximum values, minimum values, or neither.

$$\begin{aligned} f(1) &= \frac{1}{4}(1)^4 + \frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 - 1 \\ &= -\frac{11}{12} \end{aligned}$$

$$\begin{aligned} f(-1) &= \frac{1}{4}(-1)^4 + \frac{1}{3}(-1)^3 - \frac{1}{2}(-1)^2 - (-1) \\ &= \frac{5}{12} \end{aligned}$$

Begin by checking $f(1) = -\frac{11}{12}$.

Check values of the function in the neighbourhood, at points on **both** sides of $\left(1, -\frac{11}{12}\right)$.

$$f(0) = 0$$

$$\begin{aligned} f(2) &= \frac{1}{4}(2)^4 + \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 - 2 \\ &= 4 + \frac{8}{3} - 2 - 2 \\ &= \frac{8}{3} \end{aligned}$$

Since both functional values are larger than $f(1) = -\frac{11}{12}$, that value must be a relative minimum. This can be confirmed by looking at the slope of the curve on either side.

$$\begin{aligned} f'(0) &= (0)^3 + (0)^2 - (0) - 1 \\ &= -1 \\ &< 0 \end{aligned}$$

On the left, the graph is falling.

$$\begin{aligned} f'(2) &= (2)^3 + (2)^2 - (2) - 1 \\ &= 9 \\ &> 0 \end{aligned}$$

On the right, the graph is rising.

Since the graph falls to $\left(1, -\frac{11}{12}\right)$ and then rises, the point is a local minimum.

Now check $f(-1) = \frac{5}{12}$.

Check values of the function in the neighbourhood, at points on **both** sides of $\left(-1, \frac{5}{12}\right)$.

$$f(0) = 0$$

This value is smaller than $f(-1) = \frac{5}{12}$.

$$f(-2) = \frac{1}{4}(-2)^4 + \frac{1}{3}(-2)^3 - \frac{1}{2}(-2)^2 - (-2)$$
$$= 4 - \frac{8}{3} - 2 + 2$$
$$= \frac{4}{3}$$

This value is larger than $\frac{5}{12}$. Perhaps the stationary point $\left(-1, \frac{5}{12}\right)$ is neither a maximum nor a minimum. Confirm this by looking at the slope of the curve on either side.

$$f'(0) = 0^3 + 0^2 - 0 - 1$$
$$= -1$$
$$< 0$$

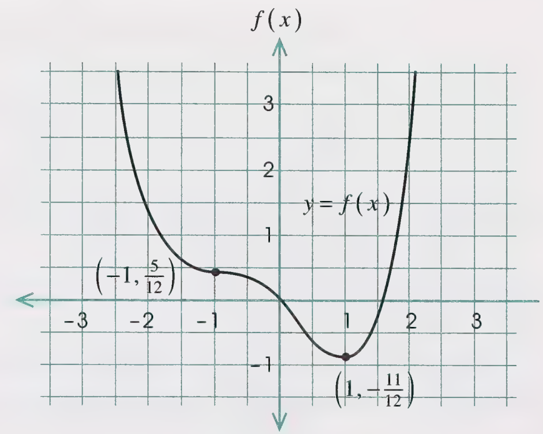
On the right, the graph is falling.

$$f'(-2) = (-2)^3 + (-2)^2 - (-2) - 1$$
$$= -8 + 4 - 2 - 1$$
$$= -7$$
$$< 0$$

On the left, the graph is falling.

Since the graph falls on both sides of $\left(-1, \frac{5}{12}\right)$, this point is neither a local minimum nor a local maximum.

Confirm your findings by looking at the graph.



There are no maximum values; $f(1) = -\frac{11}{12}$ is an absolute minimum.

Use this procedure in question 4.

4. Find the extreme values of $y = \frac{2}{3}x^3 - x^2 - 24x + 2$.

Check your answers by turning to the Appendix.

Example 3

Find the maximum or minimum points on the graph of $y = 3(x-2)^{\frac{2}{3}} + 1$. Sketch the graph.

Solution

Differentiate.

$$\begin{aligned}\frac{dy}{dx} &= 3\left(\frac{2}{3}\right)(x-2)^{\frac{2}{3}-1}(1)+0\\ &= 2(x-2)^{-\frac{1}{3}}\\ &= \frac{2}{(x-2)^{\frac{1}{3}}}\end{aligned}$$

The derivative is never 0; therefore, there are no stationary points. However, the slope is infinite at $x = 2$. You must check the point on the graph associated with this critical value of x.

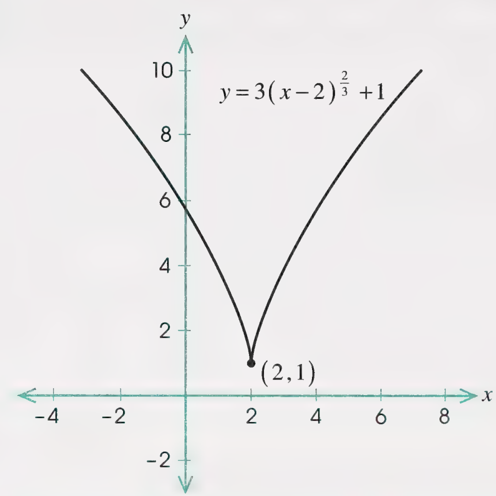
$$\begin{aligned}\text{When } x = 2,\ y &= 3(2-2)^{\frac{2}{3}}+1\\ &= 3(0)+1\\ &= 1\end{aligned}$$

Test the slope of the curve on either side of $(2, 1)$.

If $x < 2$, then $x - 2 < 0$ and $(x-2)^{\frac{1}{3}} < 0$. Therefore, $\frac{dy}{dx} = \frac{2}{(x-2)^{\frac{1}{3}}}$ must also be negative. Thus, the curve falls.

If $x > 2$, then $x - 2 > 0$ and $(x-2)^{\frac{1}{3}} > 0$. Therefore, $\frac{dy}{dx} = \frac{2}{(x-2)^{\frac{1}{3}}}$ must also be positive. Thus, the curve rises.

Since the curve changes from falling to rising at $(2, 1)$, this point must be a turning point—a local minimum. This is verified by the graph. Notice that $(2, 1)$ is also an absolute minimum.



The point $(2,1)$ is called a **cusp**. A cusp is a point where two tangents merge into a single tangent. A tangent to the curve to the left of $(2,1)$ and a tangent to the curve to the right of $(2,1)$ each become closer to the vertical as they are moved near $(2,1)$. At $(2,1)$ those tangents would coincide.

Example 4

Find the extreme values of $f(x)=\sqrt{x+3}$.

Solution

$$f(x)=\sqrt{x+3}$$
$$=(x+3)^{\frac{1}{2}}$$

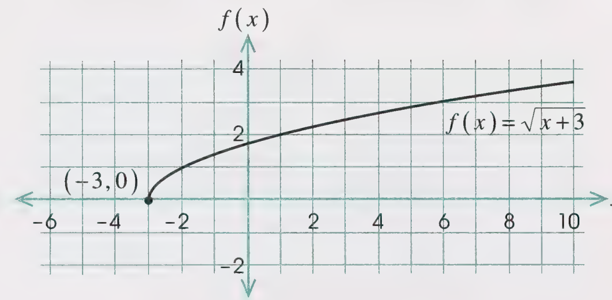
$$f'(x)=\frac{1}{2}(x+3)^{\frac{1}{2}-1}(1)$$
$$=\frac{1}{2}(x+3)^{-\frac{1}{2}}$$
$$=\frac{1}{2(x+3)^{\frac{1}{2}}}$$

The slope becomes infinite as $x \to -3^{+}$. Also, this function is only defined on the interval $[-3,\infty)$, since square roots of negative values are non-real.

Therefore, $x=-3$ is a critical value.

$$f(-3)=\sqrt{-3+3}$$
$$=0$$

Since this is an endpoint, simply check the slope on the right. Because $f'(x)>0$ for $x>-3$, the graph always rises. Therefore, point $(-3,0)$ is a minimum point, and $f(-3)=0$ is both a relative and absolute minimum.



5. For each of the following functions, determine the extreme values. Sketch each graph.

   a. $f(x)=(x-2)^4+1$

   b. $f(x)=\dfrac{x^2}{1-x}$

   c. $f(x)=-x^2-2x$, where $-2 \le x \le 1$

   d. $y=x(2x-1)^{\frac{1}{2}}$

Check your answers by turning to the Appendix.

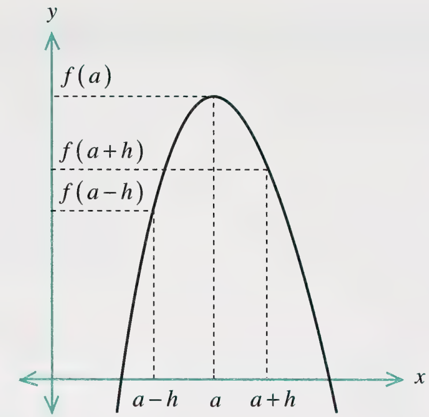
Did you know that Snell's Laws of Refraction can be derived using calculus and minimizing the time for light to travel from one medium to another? You may wish to read about the refraction of light, and investigate the formulas involved.

Activity 3: Necessary and Sufficient Conditions

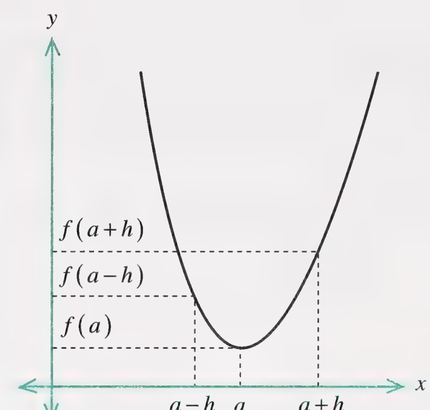
In your work in the preceding activities, you noticed that maximum and minimum values occur in different situations. Two questions arise. What conditions are necessary for extreme values to occur? What conditions are sufficient for you to conclude that the graph of a particular function will have maxima or minima?

What is necessary for a function to have a maximum or minimum at $x = a$, if the function is defined on an interval extending on both sides of $x = a$?

A function *f* has a relative or local maximum at $x = a$, if $f(a) \geq f(a \pm h)$ for all values of *h* sufficiently close to 0.



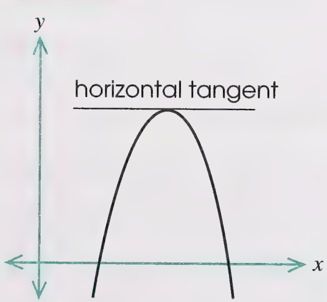
A function *f* has a relative or local minimum at $x = a$, if $f(a) \leq f(a \pm h)$ for all values of *h* sufficiently close to 0.

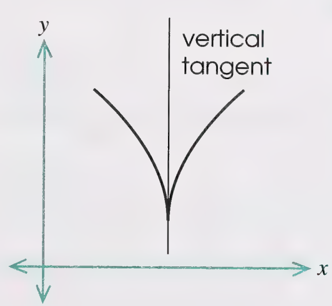


You can also conclude that a continuous function *f* has a relative maximum at $x = a$, if, at $x = a$, the function ceases to increase and begins to decrease; that is, the slope of the curve changes from positive to negative.

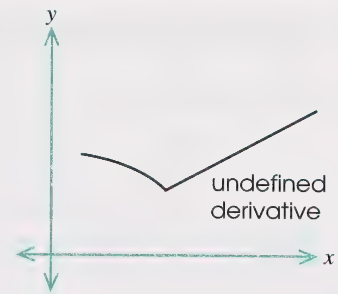
A continuous function *f* has a relative minimum at $x = a$, if, at $x = a$, the function ceases to decrease and begins to increase; that is, the slope of the curve changes from negative to positive.

Often, but not always, at those points of transition, the slope of the curve is zero or infinite; that is, the tangent line is either horizontal or vertical.

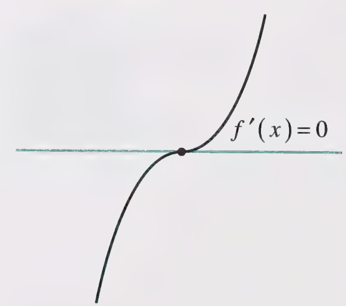


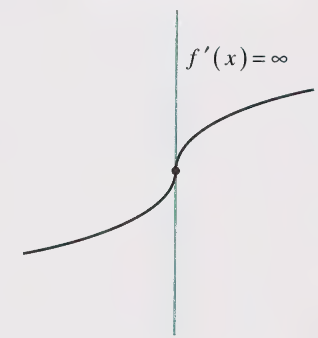


However, the slope at a maximum or minimum is not necessarily zero or infinite. At the minimum point, as shown in the following diagram, the slope is undefined. The slope of the curve, as you approach the point from the left, is not the same as the slope of the curve as you approach that point from the right.

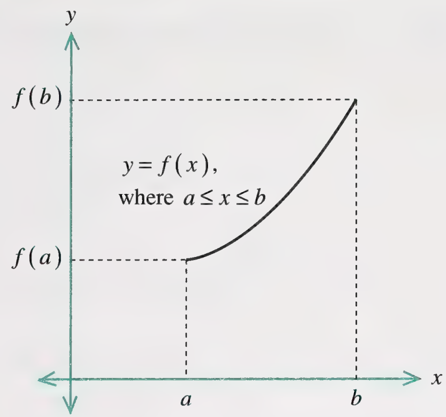


Also, finding that the slope at a point is zero or infinite is not sufficient to assert that a maximum or minimum exists. In the following diagrams, maximum or minimum values do not exist.

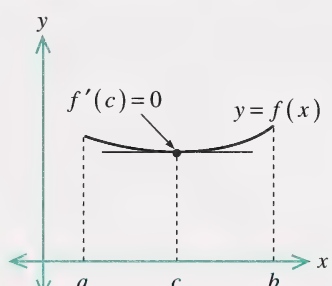




In all of the preceding discussion, it was assumed that the maximum or minimum values did not occur at the endpoints of an interval. Remember, you must investigate the endpoints of the interval on which the function is defined for extreme values. In the following diagram, $f(a)$ is a minimum and $f(b)$ is a maximum.



Don't despair! There is one thing you can say for certain. If a function *f* is defined for the interval $[a, b]$ and has a minimum or maximum at $x = c$, where $a < c < b$, and if the derivative $f'(c)$ exists and is finite, then $f'(c) = 0$. Many continuous functions, such as polynomial functions, behave **nicely** like this.



In summary, when determining extreme values of a function *f*, determine critical values of *x*: values where f' is zero, infinite, or undefined. These locations warrant further investigation. To determine whether an extreme value is a minimum or maximum, examine the values of the function on each side if the function is defined there, or examine the slope on each side.

Note: Don't forget to check endpoints!

Example 1

For $f(x) = x^4 - 5x^2 + 4$, find the critical values of *x*. Determine where the function increases and decreases. Locate maximum and/or minimum points. Use this information to sketch the graph.

Solution

Use the derivative to find the critical values of *x*.

$$\begin{aligned} f'(x) &= 4x^3 - 5(2x) + 0 \\ &= 4x^3 - 10x \end{aligned}$$

Since the derivative is defined and finite for all real values of *x*, the critical values occur when $f'(x) = 0$ (the stationary point).

$$\begin{aligned} \therefore\ 4x^3 - 10x &= 0 \\ x\left(4x^2 - 10\right) &= 0 \end{aligned}$$

$$x = 0 \quad \text{or} \quad 4x^2 - 10 = 0$$
$$4x^2 = 10$$
$$x^2 = \frac{10}{4}$$
$$x = \pm\frac{\sqrt{10}}{2}$$

The critical values are $x = 0$ and $x = \pm\frac{\sqrt{10}}{2}$.

Next, locate the stationary points.

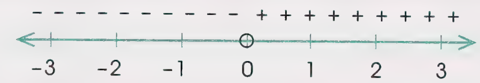
$$f(0) = (0)^4 - 5(0)^2 + 4$$
$$= 4$$

$$f\left(\pm\frac{\sqrt{10}}{2}\right) = \left(\pm\frac{\sqrt{10}}{2}\right)^4 - 5\left(\pm\frac{\sqrt{10}}{2}\right)^2 + 4$$
$$= \frac{100}{16} - 5\left(\frac{10}{4}\right) + 4$$
$$= -\frac{9}{4}$$

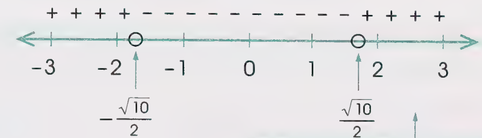
Therefore, the stationary points are $\left(-\frac{\sqrt{10}}{2}, -9\right)$, $(0, 4)$, and $\left(\frac{\sqrt{10}}{2}, -9\right)$.

Determine the intervals where the function increases and decreases. The curve rises when $f'(x) > 0$ and falls when $f'(x) < 0$.

Sign of x

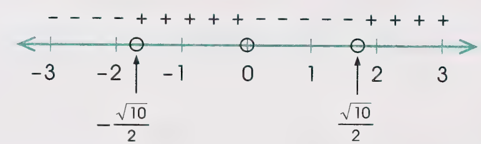


Sign of $4x^2 - 10$



$$4x^2 - 10 > 0$$
$$x^2 > \frac{10}{4}$$
$$x > \frac{\sqrt{10}}{2} \text{ or } x < -\frac{\sqrt{10}}{2}$$

Sign of $x\left(4x^2 - 10\right)$



The function increases on the interval $\left(-\frac{\sqrt{10}}{2}, 0\right) \cup \left(\frac{\sqrt{10}}{2}, \infty\right)$.

The function decreases on the interval $\left(-\infty, -\frac{\sqrt{10}}{2}\right) \cup \left(0, \frac{\sqrt{10}}{2}\right)$.

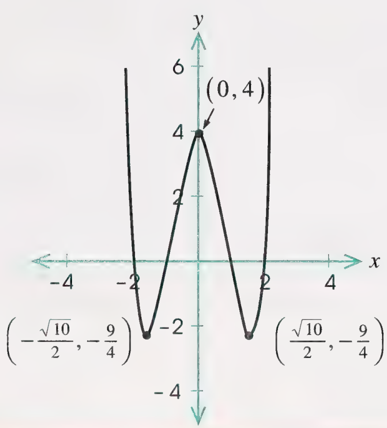
Therefore, the point $\left(-\frac{\sqrt{10}}{2}, -\frac{9}{4}\right)$ is a local minimum because the graph falls and then rises. The point $(0, 4)$ is a local maximum because the graph rises and then falls. The point $\left(\frac{\sqrt{10}}{2}, -\frac{9}{4}\right)$ is a local minimum because the graph falls and then rises.

Before sketching this curve, determine the x-intercepts.

$$x^4 - 5x^2 + 4 = 0$$
$$\left(x^2 - 4\right)\left(x^2 - 1\right) = 0$$
$$(x-2)(x+2)(x-1)(x+1) = 0$$

The x-intercepts are ± 1 and ± 2.

Now, sketch the graph of the function.



Example 2

Find the critical values of x for $f(x) = \frac{\sqrt{x+1}}{x}$. Determine where the function increases and decreases. Locate maximum and minimum points. Determine all the asymptotes (if any). Use this information to sketch the graph.

Solution

The graph of the function has a vertical asymptote at $x = 0$.

$$\lim_{x \to +\infty} \frac{\sqrt{x+1}}{x} = \lim_{x \to +\infty} \frac{\left[x^2\left(\frac{1}{x} + \frac{1}{x^2}\right)\right]^{\frac{1}{2}}}{x}$$
$$= \lim_{x \to +\infty} \frac{x\left[\frac{1}{x} + \frac{1}{x^2}\right]^{\frac{1}{2}}}{x}$$
$$= [0 + 0]^{\frac{1}{2}}$$
$$= 0$$

Therefore, the horizontal asymptote is $y = 0$.

Determine any critical values of x.

Differentiate the function using the quotient rule.

$$f'(x) = \frac{vu' - uv'}{v^2}$$

$$u = (x+1)^{\frac{1}{2}} \qquad v = x$$
$$u' = \frac{1}{2}(x+1)^{-\frac{1}{2}} \qquad v' = 1$$

$$\begin{aligned} f'(x) &= \frac{x\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}} - (x+1)^{\frac{1}{2}}(1)}{x^2} \\ &= \frac{\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}}\left[x - 2(x+1)\right]}{x^2} \\ &= \frac{(x+1)^{-\frac{1}{2}}(x - 2x - 2)}{2x^2} \\ &= \frac{(x+1)^{-\frac{1}{2}}(-x-2)}{2x^2} \end{aligned}$$

If $f'(x) = 0$, then $-x - 2 = 0$

$$x = -2$$

However, this is not possible since $f(x) = \frac{\sqrt{x+1}}{x}$ implies $x + 1 > 0$ or $x > -1$. Therefore, there are no stationary points. Also, there are no finite points where the slope is infinite, as the original function is undefined at $x = 0$. However, you must check the curve at $x = -1$, because that value is the endpoint of the interval $[-1, 0) \cup (0, \infty)$ on which the function is defined.

$$\begin{aligned} f(-1) &= \frac{\sqrt{-1+1}}{-1} \\ &= 0 \end{aligned}$$

Test the point $(-1, 0)$; is it a minimum or maximum? Determine where the function increases and where it decreases.

$$f'(x) = \frac{(x+1)^{-\frac{1}{2}}(-x-2)}{2x^2}$$

The only part of this expression which could be positive or negative is $(-x-2)$; the other parts are always positive. Therefore, the sign of the derivative depends solely on $(-x-2)$.

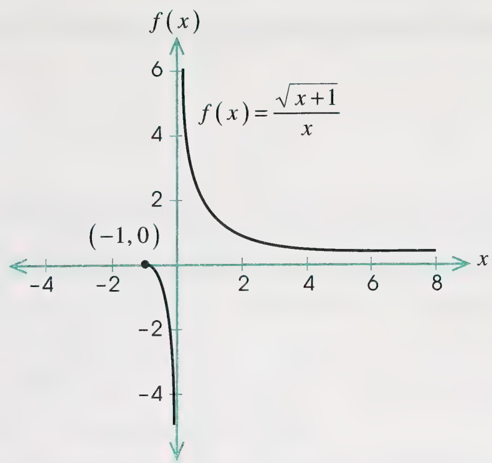
The graph falls when $f'(x) < 0$.

When $-x - 2 < 0$

$$\begin{aligned} -x &< 2 \\ x &> -2 \end{aligned}$$

But this is always the case! The function decreases throughout its domain: $[-1, 0) \cup (0, \infty)$.

The point $(-1, 0)$ must be a local maximum since the curve falls to its right. Now sketch the curve.



Do the following questions.

1. Determine the absolute minimum and the absolute maximum of the function $f(x)=x^2-6x-3$, where $1 \le x \le 4$.

2. Find the critical values of x for $f(x)=(x-3)^3(3x-2)^2$.

3. Show that the polynomial function $f(x)=x^5+x$ has no stationary points, and therefore, no relative maxima and minima.

4. Verify that $f(x)=-3x^{\frac{2}{3}}$ has a relative maximum at $x=0$.

5. Find the value of b if $f(x)=-2x^2+bx-7$ has a local maximum at $x=4$.

6. Find the minimum and maximum values of $f(x)=\frac{x^3+4}{x^2}$; then sketch the graph of the function.

7. Sketch the graph of a function for which the following is true.

- $f(-3)=f(3)=17$
- $f(-2)=f(2)=-8$
- $f(0)=8$
- $f'(-2)=f'(0)=f'(2)=0$
- $f'(x)>0$ on the interval $(-2,0)\cup(2,\infty)$
- $f'(x)<0$ on the interval $(-\infty,-2)\cup(0,2)$

8. If (a,b) is a stationary point of $y=f(x)$, then it is also a stationary point of $y=[f(x)]^n$. Prove this statement.

Check your answers by turning to the Appendix.

In Section 2: Extra Help, you were given the option to view the video titled *Derivatives and Graph Sketching* from the *Catch 31* series, ACCESS Network. If you have not watched this video segment yet, then watch the first half of it. This video segment discusses curve sketching in a historical context. It reviews fundamental procedures such as using tables of values, finding intercepts, and using slope to determine the intervals where a function increases, decreases, or has zero slope. Techniques for determining maximum and minimum values are discussed. Elements such as dicontinuities, vertical slopes, and cusps are reviewed. This video is available from the Learning Resources Distributing Centre.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Using what you have studied in Mathematics 30 concerning graphing polynomials will make locating maxima and minima of these functions easier. You should be able to quickly determine where a graph rises and falls by using the curve's stationary points.

Recall that the graph of the polynomial function $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n$ rises to the right as x increases without bound, provided the leading coefficient $a_0>0$. The graph falls to the right as x increases without bound when $a_0<0$. In the examples that follow, you will see why this is the case, and how you can use this fact, together with the relative positions of stationary points, to graph polynomials.

Example 1

For $f(x)=(x+1)^2(x-2)$, determine the stationary points, locate the intervals where the function increases and decreases, and sketch the curve.

Solution

Find f' and solve $f'(x)=0$ to determine the stationary points.

Use the product rule.

$$\begin{aligned}
f'(x) &= (x+1)^2\frac{d}{dx}(x-2)+(x-2)\frac{d}{dx}(x+1)^2 \\
&= (x+1)^2(1)+(x-2)(2)(x+1)(1) \\
&= (x+1)\left[(x+1)+2(x-2)\right] \\
&= (x+1)[x+1+2x-4] \\
&= (x+1)(3x-3) \\
&= 3(x+1)(x-1)
\end{aligned}$$

When $f'(x)=0$, $3(x+1)(x-1)=0$. The critical values of x are $x=1$ and $x=-1$.

Find the stationary points.

$$\begin{aligned} f(-1) &= (-1+1)^2(-1-2) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(1) &= (1+1)^2(1-2) \\ &= 4(-1) \\ &= -4 \end{aligned}$$

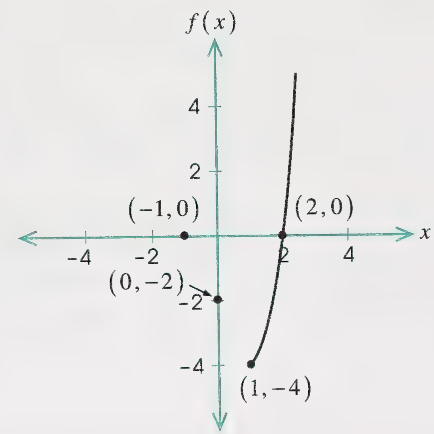
The stationary point farthest to the right is $(1,-4)$. Investigate the slope of the curve to the right of that point; that is, when $x>1$. In this interval, the derivative $f'(x)=3(x+1)(x-1)$ must be positive, because both binomial factors are positive. The graph rises in the interval $(1,\infty)$. This makes sense from what you know from Mathematics 30; if the sign of the leading coefficient is positive, the curve rises on the right.

$$f(x)=+(x+1)^2(x-2)$$

You will also find the intercepts useful.

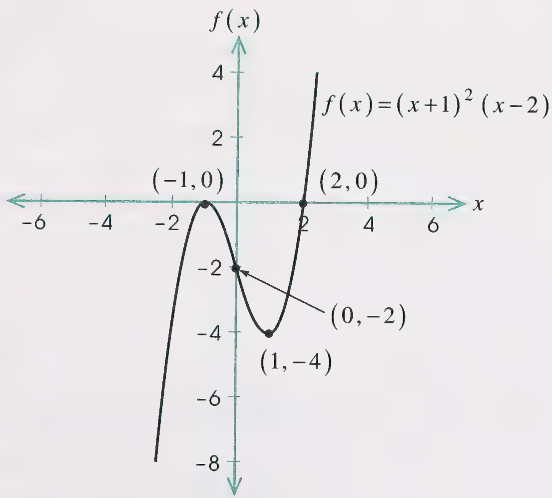
The x-intercepts are automatic; from the factors of f, they are $x=-1$ and $x=2$. Since $f(0)=(0+1)^2(0-2)=-2$, the point on the y-axis is $(0,-2)$. You should now have all the information you need to graph the function.

Begin by plotting the stationary points and the intercepts; show the graph rising on the right beyond $(1,-4)$.



The rest is a simple matter of joining the points—the curve is continuous—and deciding what happens to the left of the stationary point $(-1,0)$. If you look at the derivative, its sign is positive since both binomial factors of $3(x+1)(x-1)$ are negative: $(-)\times(-)=+$. The graph rises in the interval $(-\infty,-1)$.

Complete the graph.



There is a relative maximum at $(-1,0)$ and a relative minimum at $(1,-4)$. The function increases in the interval $(-\infty,-1)\cup(1,\infty)$. The function decreases in the interval $(-1,1)$.

All polynomial functions, as in Example 1, are similarly sketched. Look at an example with a negative leading coefficient.

Example 2

Sketch and describe $f(x)=-2(x-3)^2(x+3)^3+1$.

Solution

Find f' and solve $f'(x)=0$ to determine the stationary points.

Use the product rule to differentiate.

$$\begin{aligned} f'(x) &= -2(x-3)^2\frac{d}{dx}(x+3)^3+(x+3)^3\frac{d}{dx}(-2)(x-3)^2+0 \\ &= -2(x-3)^2(3)(x+3)^2+(x+3)^3(-2)(2)(x-3) \\ &= -2(x-3)(x+3)^2[3(x-3)+2(x+3)] \\ &= -2(x-3)(x+3)^2[3x-9+2x+6] \\ &= -2(x-3)(x+3)^2(5x-3) \end{aligned}$$

When $f'(x)=0$, then $-2(x-3)(x+3)^2(5x-3)=0$. Therefore, the critical values are $x=3$, $x=-3$, and $x=\frac{3}{5}$.

Find the stationary points.

$$\begin{aligned} f(3) &= -2(3-3)^2(3+3)^3+1 \\ &= 1 \end{aligned}$$

$$f(-3) = -2(-3-3)^2(-3+3)^3 + 1$$
$$= 1$$

$$f\left(\frac{3}{5}\right) = -2\left(\frac{3}{5}-3\right)^2\left(\frac{3}{5}+3\right)^3 + 1$$
$$= -536.477\,12$$

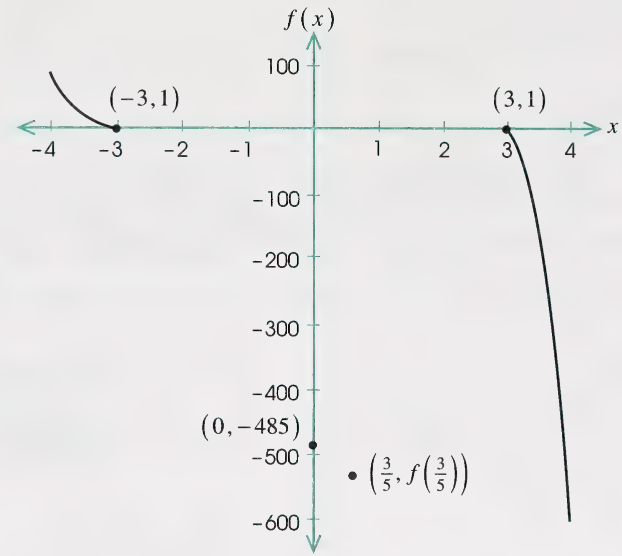
Beyond $(3, 1)$, the farthest stationary point on the right, the sign of $f'(x) = -2(x-3)(x+3)^2(5x-3)$ must be negative, as each binomial factor is positive but the leading coefficient is negative. This is consistent with what you studied in Mathematics 30—if the leading coefficient is negative, the graph falls on the right of the last stationary point.

To the left of $(-3, 1)$, the sign of $f'(x) = -2(x-3)(x+3)^2(5x-3)$ must be negative, as each binomial factor is negative and the leading coefficient is negative. The product of five negative factors is negative. The graph falls for x in the interval $(-\infty, -3)$.

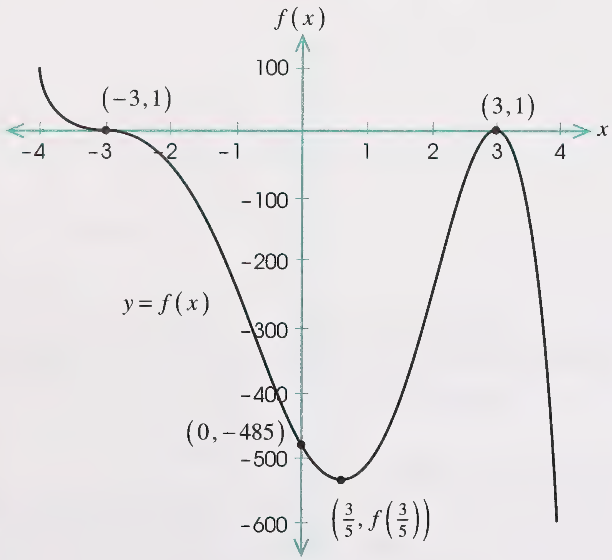
Find the y-intercept.

$$f(0) = -2(0-3)^2(0+3)^3 + 1$$
$$= -2(9)(27) + 1$$
$$= -485$$

Now plot the stationary points and the y-intercept. Show the graph falling for the intervals $(3, \infty)$ and $(-\infty, -3)$.



Now draw a smooth curve through the points. Notice that $f(-3)$ is neither a minimum nor a maximum.



From the graph, $f\left(\frac{3}{5}\right)$ is a relative minimum; $f(3)=1$ is a relative maximum. The graph falls for values of x in $(-\infty,-3)\cup\left(-3,\frac{3}{5}\right)\cup(3,\infty)$; the graph rises for the interval $\left(\frac{3}{5},3\right)$.

In Example 3, you will be given the locations of the stationary points, the intercepts, and the curve behaviour on the right and left of the points.

Example 3

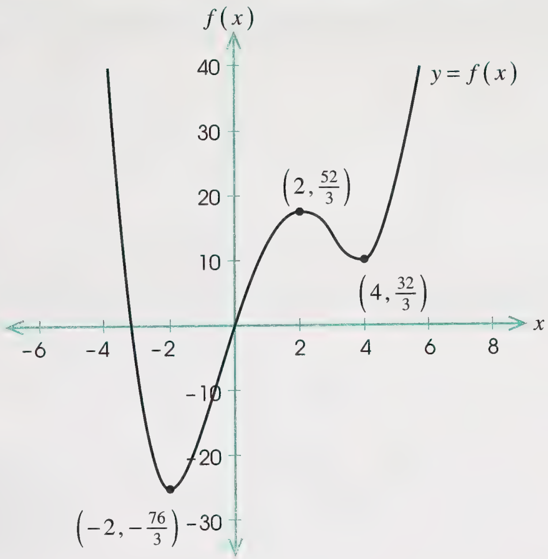
Sketch a polynomial function for which each of the following is true.

- $f(0)=0$
- $f'(-2)=f'(2)=f'(4)=0$
- $f(-2)=-\frac{76}{3},\ f(2)=\frac{52}{3},\ f(4)=\frac{32}{3}$
- The graph falls when $x<-2$.
- The leading coefficient is positive.

State the maximum and minimum values. Where is the function decreasing? Where is the function increasing?

Solution

Begin by plotting the stationary points: $\left(-2,-\frac{76}{3}\right)$, $\left(2,\frac{52}{3}\right)$, and $\left(4,\frac{32}{3}\right)$. Since the leading coefficient is positive, the graph rises for $x>4$. Draw a smooth curve through the points, making certain the curve passes through the origin. The graph should fall when $x<-2$.



The graph should have a relative minimum of $-\frac{76}{3}$ at $x=-2$, a relative minimum of $\frac{32}{3}$ at $x=4$, and a relative maximum of $\frac{52}{3}$ at $x=2$. The function decreases for x in the interval $(-\infty,-2)\cup(2,4)$. The function increases for x in the interval $(-2,2)\cup(4,\infty)$.

1. Sketch the graph of $f(x)=-(x-2)(x+4)$. What is the maximum and/or minimum point? Where does the curve rise and where does it fall?

2. Graph $f(x)=2x^3-3x^2$. What is the maximum point and/or the minimum point? Where does the curve rise and where does it fall?

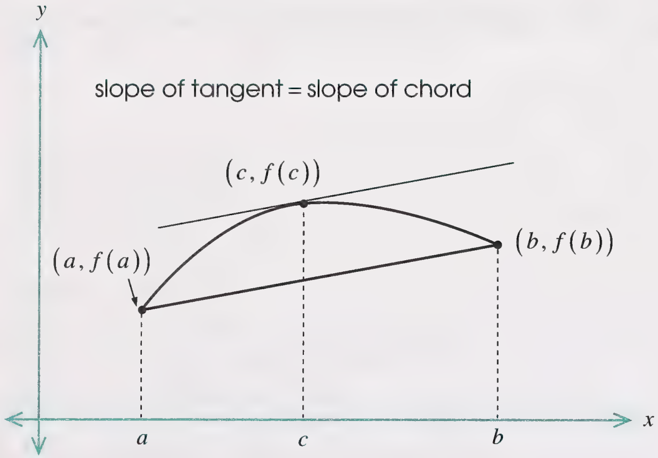
3. Sketch the graph of a polynomial function for which the following is true:

   - $f'(-2)=f'(2)=0$
   - $f(-2)=-16$ and $f(2)=16$
   - The leading coefficient is negative.
   - $f'(x)<0$ when $x<-2$
   - $f(0)=f\left(\pm 2\sqrt{3}\right)=0$

Check your answers by turning to the Appendix.

Enrichment

The following discussion introduces the Mean Value Theorem. This theorem deals with a function *f* which is continuous on the interval $[a,b]$ and which has a finite derivative at each point in (a,b); that is, the tangents to the curve exist and are non-vertical for all points in the open interval (a,b). Tangents, in fact, could be vertical at either $x=a$ or $x=b$, or both, without invalidating the results of the theorem. Consider the following diagram.



The Mean Value Theorem states that there exists at least one point c between $x = a$ and $x = b$ where the slope of the tangent is equal to the slope of the chord joining $(a, f(a))$ and $(b, f(b))$; that is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In the examples that follow, you will be asked to find that point in the interval where the tangent is parallel to the chord.

Example 1

Given $f(x) = x^2 - 3x$, and $a = 1$, and $b = 3$, determine c in the interval (a, b), where $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Solution

First, find the slope of the chord.

$$f(1) = 1^2 - 3(1) \qquad f(3) = 3^2 - 3(3)$$
$$= -2 \qquad\qquad = 0$$

$$\begin{aligned}\text{slope of the chord} &= \frac{f(3) - f(1)}{3 - 1} \\ &= \frac{0 - (-2)}{2} \\ &= 1\end{aligned}$$

Next, find the slope of a tangent by differentiating the function.

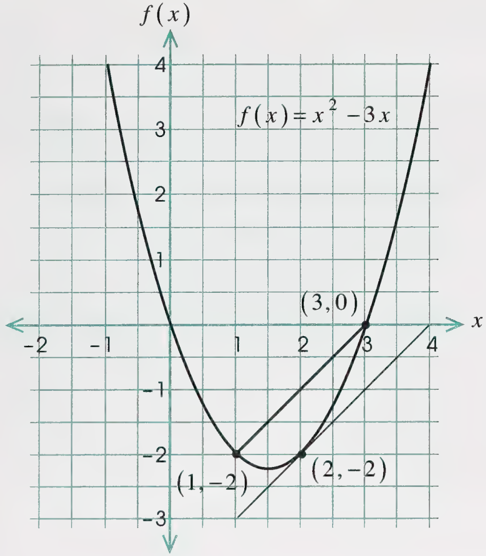
$$f'(x) = 2x - 3$$

If the tangent at $x = c$ is parallel to the chord, $f'(c) =$ slope of the chord.

$$\begin{aligned}2c - 3 &= 1 \\ 2c &= 4 \\ c &= 2\end{aligned}$$

$$f(2)=2^2-3(2)$$
$$=-2$$

The tangent at $(2,-2)$ should be parallel to the chord.

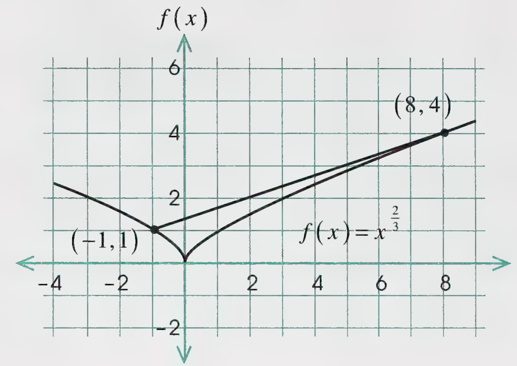


In Example 2 you will find the value of c in an interval, where the given function is not differentiable at every point.

Example 2

Given $f(x)=x^{\frac{2}{3}}$ and $a=-1$ and $b=8$, determine c in the interval (a,b), where $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Solution



First, find the slope of the chord.

$$f(-1)=(-1)^{\frac{2}{3}}$$
$$=1$$

$$f(8)=(8)^{\frac{2}{3}}$$
$$=\left(8^{\frac{1}{3}}\right)^2$$
$$=2^2$$
$$=4$$

$$\text{slope of the chord} = \frac{f(8)-f(-1)}{8-(-1)}$$
$$= \frac{4-1}{8-(-1)}$$
$$= \frac{1}{3}$$

Next, find the slope of a tangent by differentiating the function.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

The derivative is undefined at $x = 0$. Since the function is not differentiable at $x = 0$, the conditions of the Mean Value Theorem are not met. However, try to find if there is a tangent parallel to the chord. If the tangent at $x = c$ is parallel to the chord, $f'(c) =$ slope of the chord.

$$\frac{2}{3}c^{-\frac{1}{3}} = \frac{1}{3}$$
$$c^{-\frac{1}{3}} = \frac{1}{2}$$
$$c^{\frac{1}{3}} = 2$$
$$c = 8$$

But this value does not lie in the interval $(-1, 8)$. There is no point strictly between $x = -1$ and $x = 8$ where the tangent is parallel to the chord, since the function is not differentiable at $x = 0$.

1. Given $f(x) = x^3$ and $a = -1$ and $b = 2$, determine c in the interval (a, b), where $f'(c) = \frac{f(b)-f(a)}{b-a}$.

2. Given $f(x) = x^{\frac{2}{3}}$ and $a = -1$ and $b = +1$, show that there is no c in the interval (a, b), where $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Check your answers by turning to the Appendix.

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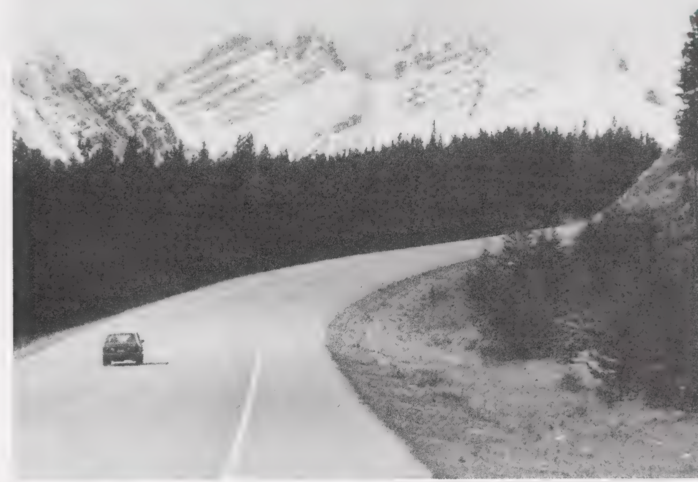
Conclusion

In this section, you discovered how to use the first derivative as a tool in curve sketching. You should now be able to do the following:

- identify, from a sketch, locations at which the first derivative is zero or undefined
- relate the zeros of the derivative function to the critical points on the original curve
- explain circumstances where maximum and minimum values occur when $f'(x)$ is not zero
- explain why $f'(x) = 0$ will not necessarily yield a maximum or minimum
- verify whether a point is a maximum or minimum
- use the first derivative to find maximum and minimum points to aid in sketching graphs, and compare these sketches to calculator- or computer-generated plots of the same function
- explain the differences between local and absolute maxima and minima
- explain how the sign of the first derivative indicates whether a curve is rising or falling
- verify whether a point is a maximum or minimum

If you are not certain about any of these concepts or procedures, review the examples from the activities.

The first derivative is fundamental in determining precisely where a curve rises or falls. The tools of calculus are much more accurate in analysing the twists and turns of a graph than the human eye. Seeing is not always believing. Have you experienced the illusion, when travelling in the mountains, that the car in which you are riding is going downhill even though you know, by the way the engine is labouring, that you must still be climbing?



Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 4: The Second Derivative

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Railroad tracks along mountain valleys curve in and out, winding their way around outcrops of rock. In the photo, the track curves inward towards the photographer. The locomotive in the distance is approaching a bend that curves in the opposite direction around the mountainside. Somewhere between the two bends, the curvature changes.

Curvature is an important feature of the graph of a function. Calculus provides tools for analysing the concavity of a curve. The sign of the second derivative tells you whether a curve opens downward, like an arched doorway, or opens upward, like a cable suspended between two supporting towers.

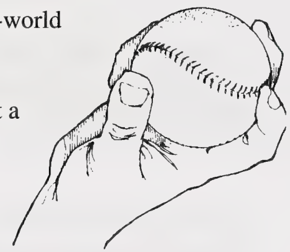
Activity 1 deals with tests for concavity, and how knowing the concavity at an extreme point will confirm whether that point is a maximum or minimum. The test for concavity, together with the Second Derivative Test for maxima and minima, are essential procedures in curve sketching.

In Activity 2, you will locate points of inflection—points of transition where the curve changes concavity.

This is the last set of concepts you will have to master. In the next section, tests for concavity will be combined with other curve-sketching techniques for a comprehensive strategy for both algebraic and trigonometric functions.

Activity 1: Concavity

In this activity you will investigate another aspect of the graph of a function, called **concavity**. Concavity is the measure of the curvature of a graph.



Concavity can be illustrated using a real-world example involving the motion of a ball.

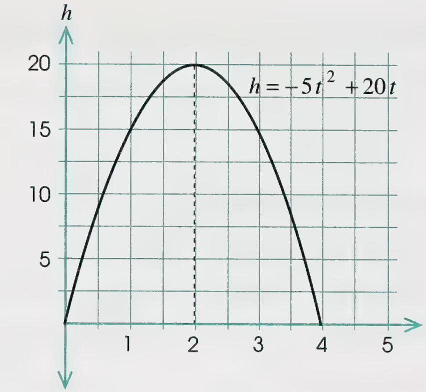
For instance, if a ball is tossed upward at a speed of 20 m/s, its height h (in metres), after t seconds, is given by

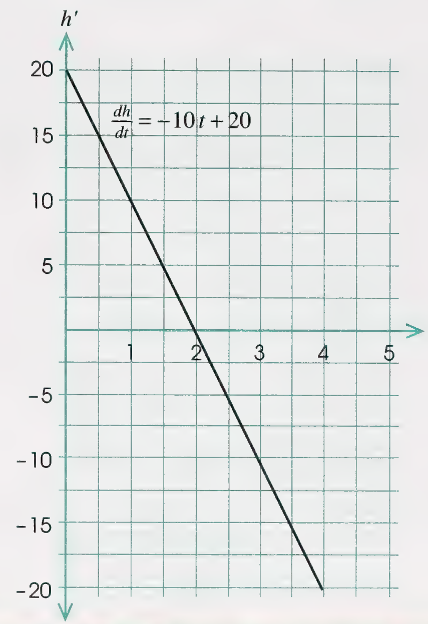
$h = -5t^2 + 20t$, where $t \geq 0$. You will recall that the slope of the graph $\frac{dh}{dt}$ is the rate of change of height with respect to time, or, the velocity of the ball.

$$\frac{dh}{dt} = -10t + 20, \text{ where } t \geq 0$$

When you draw the graphs of the original function and its derivative, the ball's speed relative to its position is apparent.

Note: The ball is in the air for only 4 s.





A comparison of the graphs reveals that the stationary point of the graph $h=-5t^2+20t$, where $t \geq 0$ occurs at $t=2$. Of course, the velocity $\frac{dh}{dt}$ at that time is 0.

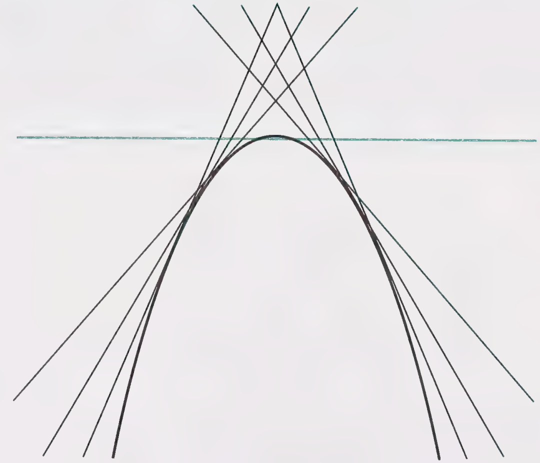
$$\text{At } t=2,\ \frac{dh}{dt} = -10(2)+20$$
$$= 0$$

When the ball rises, its velocity is positive; the graph of the original function rises in the interval $[0, 2)$. When the ball falls, its velocity is negative; the graph of the original function falls in the interval $(2, 4]$.

$$\text{Now, at } t=2,\ h=-5(2)^2+20(2)$$
$$= 20$$

Since the graph rises and then falls, the turning point $(2, 20)$ is a relative maximum. The ball rises to a maximum height of 20 m.

The original graph opens downward from $(2, 20)$. The curve is said to be **concave downward** at that point. The reason the graph is concave downward is contingent on how the slopes of tangents to that curve are changing.

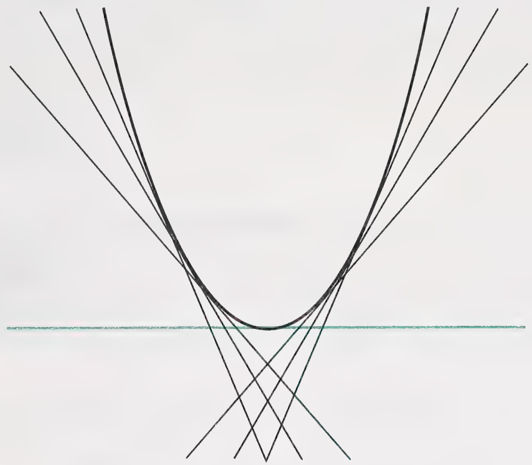


As you move from left to right along the curve, the slopes of tangent lines become less positive, they pass through zero at the stationary point, and then they become negative on the right. The original graph can be described as the curve traced out by this envelope of tangent lines.

Therefore, the fact that the curve is concave downward can be related to the second derivative $\frac{d^2h}{dt^2}=-10$. The second derivative describes how the first derivative is changing. The slope, or first derivative, is steadily decreasing. If the first derivative is velocity, then the second derivative is the change in velocity, or acceleration. The acceleration is negative because the velocity is decreasing.

You have just seen that if the second derivative is negative, the curve is concave downward. Does that mean if the second derivative is positive, the curve is concave upward?

Take a look at a sequence of tangent lines with increasing slope.



The curve traced out is concave upward. Study the next example.

Example 1

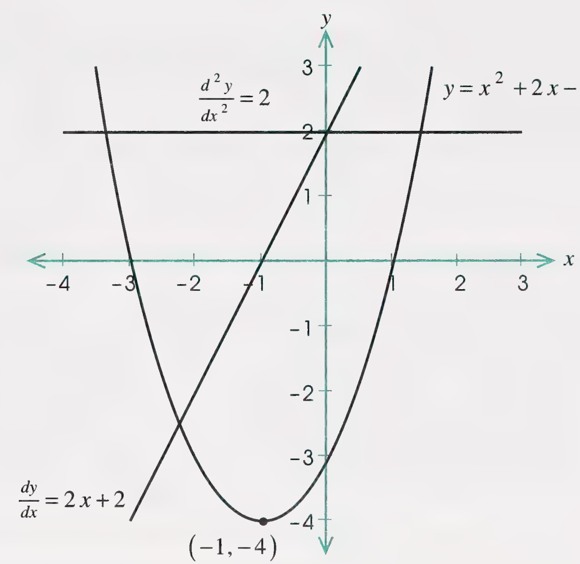
Sketch the graphs of $y = x^2 + 2x - 3$ and the first and second derivatives of the function. What does the second derivative tell you about the graph of the original function?

Solution

Find the first and second derivatives.

$$y = x^2 + 2x - 3$$
$$\frac{dy}{dx} = 2x + 2$$
$$\frac{d^2y}{dx^2} = 2$$

Sketch the graphs.



Comparing the graphs, you see that the minimum point of the graph of $y = x^2 + 2x - 3$ occurs at $(-1, -4)$. The first derivative, $\frac{dy}{dx} = 2x + 2$, is negative for $x < -1$; the graph of the original function falls in that interval. The first derivative is positive for $x > -1$; the graph of the original function rises in that interval. The slope of the original curve is 0 at its stationary point. Clearly, the slope of the original curve increases; the graph of the first derivative rises to the right.

The second derivative, $\frac{d^2y}{dx^2} = 2$, tells you at what rate the first derivative is changing; since the first derivative increases, the second derivative must be positive. Therefore, a positive second derivative must mean that a curve is concave upward!

In summary, for the function $y = f(x)$, its graph is **concave upward** in the interval where $\frac{d^2y}{dx^2} > 0$, and is **concave downward** in the interval where $\frac{d^2y}{dx^2} < 0$.

Example 2

Determine where $f(x) = x^3 - 12x + 1$ is concave upward and concave downward. Determine all stationary points. Use concavity to decide whether those stationary points are maximum values, minimum values, or neither. Sketch the original function and the first and second derivatives. How are the graphs related?

Solution

Find the first and second derivatives.

$$f(x) = x^3 - 12x + 1$$
$$f'(x) = 3x^2 - 12$$
$$f''(x) = 6x$$

The curve is concave upward when $f''(x) > 0$.

$$6x > 0$$
$$x > 0$$

Therefore, the curve is concave upward for the interval $(0, \infty)$.

The curve is concave downward when $f''(x) < 0$.

$$6x < 0$$
$$x < 0$$

Therefore, the curve is concave upward for the interval $(-\infty, 0)$.

Next, find the stationary points which occur when $f'(x)=0$.

$$\begin{aligned} 3x^2-12&=0 \\ x^2&=4 \\ x&=\pm 2 \end{aligned}$$

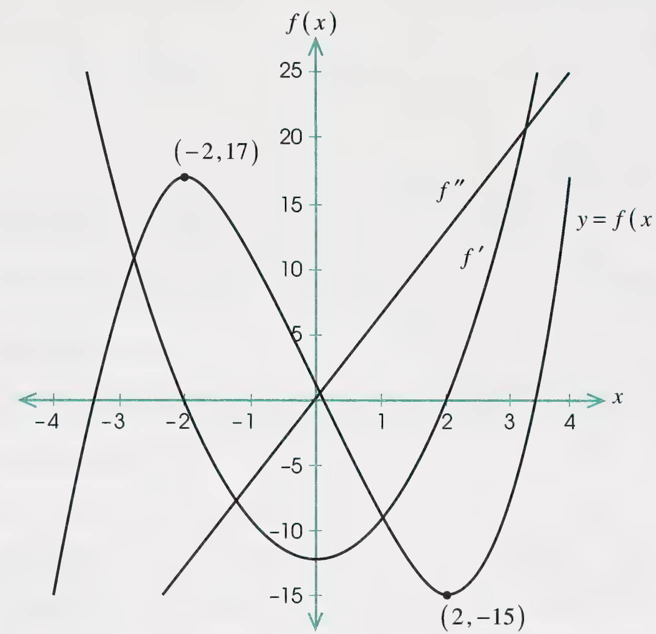
$$\begin{aligned} f(2)&=2^3-12(2)+1 \\ &=8-24+1 \\ &=-15 \end{aligned}$$

Since $(2,-15)$ lies in the interval where the curve is concave upward, $(2,-15)$ must be a relative minimum.

$$\begin{aligned} f(-2)&=(-2)^3-12(-2)+1 \\ &=-8+24+1 \\ &=17 \end{aligned}$$

Since $(-2,17)$ lies in the interval where the curve is concave downward, $(-2,17)$ must be a relative maximum.

The graph confirms these conclusions.



Notice, for $x<0$, where the original graph is concave downward, the graph of the first derivative is decreasing, and the graph of the second derivative lies below the x-axis. For $x>0$, where the original graph is concave upward, the graph of the first derivative is increasing, and the graph of the second derivative lies above the x-axis. The maximum on the original graph occurs when $f'(x)=0$ and $f''(x)<0$; the minimum occurs when $f'(x)=0$ and $f''(x)>0$.

1. For each of the following, determine the intervals where the functions are concave upward and concave downward. Find all stationary points. Use concavity to determine whether the stationary points are relative minima or maxima. Sketch each original function. For parts a., b., and c., sketch the graphs of the first and second derivatives.

   a. $f(x) = -x^2 + 4x + 5$ **b.** $f(x) = x^3 - 3x - 4$

   c. $f(x) = x + \frac{9}{x}$ **d.** $f(x) = (x-2)^4$

2. Sketch a continuous curve for which each of the following statements is true.

   - $f(-1) = 4$; $f(0) = 2$; $f(1) = 0$
   - $f'(-1) = f'(1) = 0$
   - $f'(x) > 0$ for the interval $(1, \infty) \cup (-\infty, -1)$
   - $f'(x) < 0$ for the interval $(-1, 1)$
   - $f''(x) < 0$ for the interval $(-\infty, 0)$
   - $f''(x) > 0$ for the interval $(0, \infty)$

Check your answers by turning to the Appendix.

In the preceding questions, the concavity at a stationary point determines whether that point is a relative minimum or maximum. This is known as the **Second Derivative Test**.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, then a relative maximum occurs.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, then a relative minimum occurs.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, then the Second Derivative Test fails.

Be careful not to read too much into this test. Remember, extreme values may occur when the first derivative is infinite or undefined. Clearly the Second Derivative Test will not work in those instances. In those cases, you must see if the derivative changes sign or you must check values of the function on both sides of those points.

Example 3

Use the Second Derivative Test to determine the maximum and minimum values of $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 1$.

Solution

Use the first derivative to locate the stationary points.

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 1$$

$$f'(x) = \frac{1}{3}(3x^2) - \frac{1}{2}(2x) - 6(1) + 0$$
$$= x^2 - x - 6$$

When $f'(x)=0,\quad x^2-x-6=0$

$$(x-3)(x+2)=0$$

$x-3=0$ or $x+2=0$

$x=3 \qquad x=-2$

$$\begin{aligned} f(-2) &= \tfrac{1}{3}(-2)^3-\tfrac{1}{2}(-2)^2-6(-2)+1 \\ &= -\frac{8}{3}-\frac{4}{2}+12+1 \\ &= \frac{25}{3} \end{aligned}$$

$$\begin{aligned} f(3) &= \tfrac{1}{3}(3)^3-\tfrac{1}{2}(3)^2-6(3)+1 \\ &= 9-\frac{9}{2}-18+1 \\ &= -\frac{25}{2} \end{aligned}$$

Next, find the values of the second derivative at these points.

$$f''(x)=2x-1$$

$$\begin{aligned} f''(-2) &= 2(-2)-1 \\ &< 0 \end{aligned}$$

Since the second derivative at $x=-2$ is negative, $f(-2)=\frac{25}{3}$ is a relative maximum.

$$\begin{aligned} f''(3) &= 2(3)-1 \\ &= 5 \end{aligned}$$

Since the second derivative at $x=3$ is positive, $f(3)=-\frac{25}{2}$ is a relative minimum.

3. For each of the following functions, use the Second Derivative Test whenever possible to determine maximum and minimum values.

a. $f(x)=x^2-6x+2$

b. $f(x)=-(x+2)^4+1$

c. $y=\dfrac{x^2}{x^2+4}$

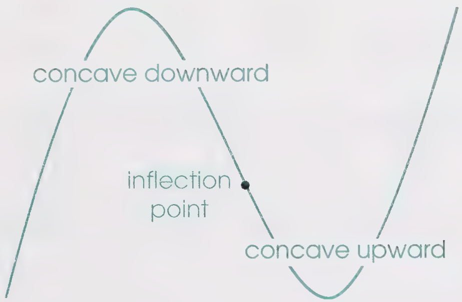
d. $y=\sqrt{x}(x-2)$

Check your answers by turning to the Appendix.

Think of the Second Derivative Test as a teacup test. If the cup **holds** tea (the second derivative is **positive**), the bottom of the cup is a **minimum**. Turn the cup upside down. The tea **pours out** (the second derivative is **negative**) and the bottom of the cup is a **maximum**.

Activity 2: Inflection Points

In the previous activity, you worked with curves for which the concavity changed from positive to negative, or negative to positive. A point where the concavity changes is called an **inflection point**.



Example 1

Determine the point of inflection of $f(x) = x^3 - 12x^2 + 12x + 4$.

Solution

Find the second derivative.

$$f'(x) = 3x^2 - 24x + 12$$
$$f''(x) = 6x - 24$$

The curve is concave downward when $f''(x) < 0$.

$$6x - 24 < 0$$
$$x < 4$$

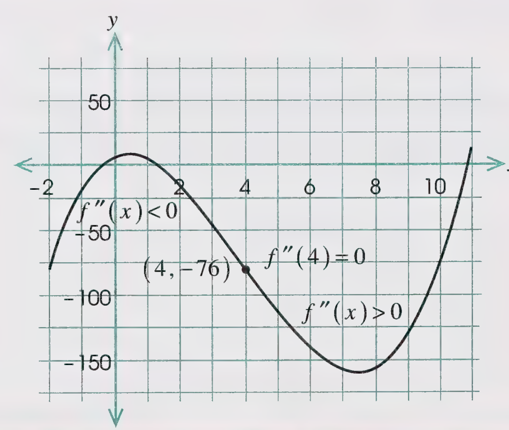
The curve is concave upward when $f''(x) > 0$.

$$6x - 24 > 0$$
$$x > 4$$

Therefore, the transition occurs at $x = 4$.

$$f(4) = (4)^3 - 12(4)^2 + 12(4) + 4$$
$$= -76$$

Notice that $f''(4) = 6(4) - 24 = 0$.

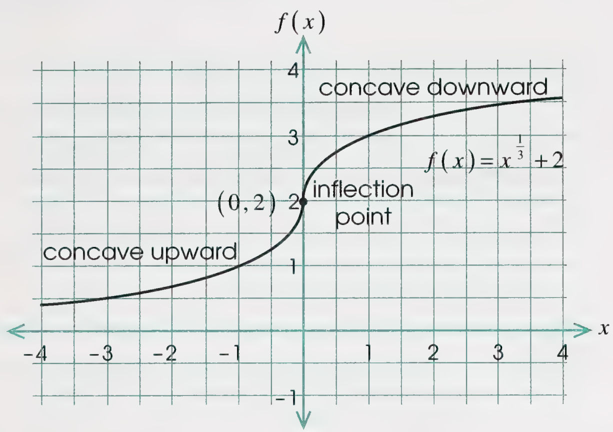


Example 1 demonstrates that a point of inflection may occur when $\frac{d^2y}{dx^2} = 0$. This makes sense as the second derivative changes from negative to positive. Is this the only possibility?

Example 2

Determine the point of inflection of $f(x) = x^{\frac{1}{3}} + 2$.

Solution



Examine the first and second derivatives.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$
$$= \frac{1}{3\left(x^{\frac{1}{3}}\right)^2}$$

The first derivative is positive for all x except 0; as $x \to 0$, $f''(x) \to \infty$. The curve rises throughout its domain.

$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$
$$= -\frac{2}{9x^{\frac{5}{3}}}$$

If $x < 0$, then $f''(x) > 0$. Thus, the curve is concave upward.
If $x > 0$, then $f''(x) < 0$. Thus, the curve is concave downward.

The curve changes concavity at $x = 0$. Since $f(0) = 0^{\frac{1}{3}} + 2 = 2$, the point of inflection occurs at $(0, 2)$.

$$f''(0) = -\frac{2}{9(0)^{\frac{5}{3}}}$$
$$= \text{undefined}$$

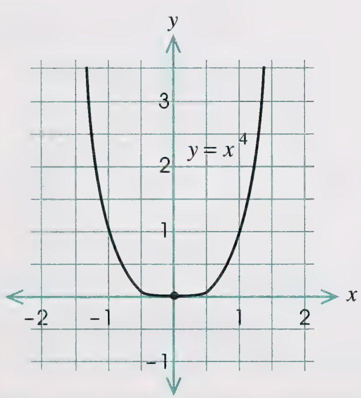
Now, $f''(x)$ becomes infinite when $x \to 0$.

A point of inflection may occur when the second derivative becomes infinite.

You should be careful when trying to locate points of inflection. You should test for concavity on both sides of the point to be certain. For instance, the second derivative of $y = x^4$ is 0 at $(0, 0)$.

$$y' = 4x^3 \qquad \text{At } x = 0,\ y'' = 12(0)^2$$
$$y'' = 12x^2 \qquad\qquad = 0$$

However, the second derivative is positive on both sides of the origin. Point $(0, 0)$ is a minimum, not a point of inflection as was originally suggested by the second derivative.



1. Graph each function. Locate the inflection points on each graph.

   a. $f(x) = (x-1)^3 - 2$

   b. $f(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - x^2 + x - 1$

   c. $f(x) = -(x-2)^{\frac{1}{5}} + 3$

2. Does $y = -x^4 + 2$ have a point of inflection? Why or why not?

Check your answers by turning to the Appendix.

In speech, the word **inflection** generally refers to a change in pitch. In mathematics, inflection refers to a change in concavity.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

What positive and negative first derivatives say about the graph of a function is different from what positive and negative second derivatives imply. Remember, if $f'(x) > 0$, the graph rises; if $f'(x) < 0$, the graph falls. If $f''(x) > 0$, the graph is concave upward; if $f''(x) < 0$, the graph is concave downward.

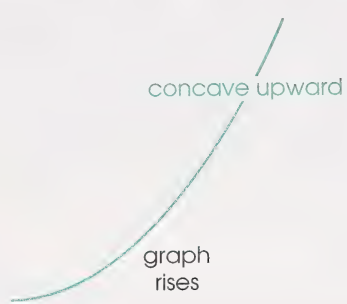
In the following example, you will look at the portion of a graph where both the first and second derivatives are positive.

Example

Sketch that portion of the graph of $y = f(x)$, where $f'(x) > 0$ and $f''(x) > 0$.

Solution

Since both derivatives are positive, the portion of the graph you will sketch will rise to the right and be concave upward.



In question 1 you will look at the other three possibilities.

1. Sketch the portion of the graph of $y = f(x)$ for which the following are true.

   a. $f'(x) > 0$ and $f''(x) < 0$
   b. $f'(x) < 0$ and $f''(x) > 0$
   c. $f'(x) < 0$ and $f''(x) < 0$

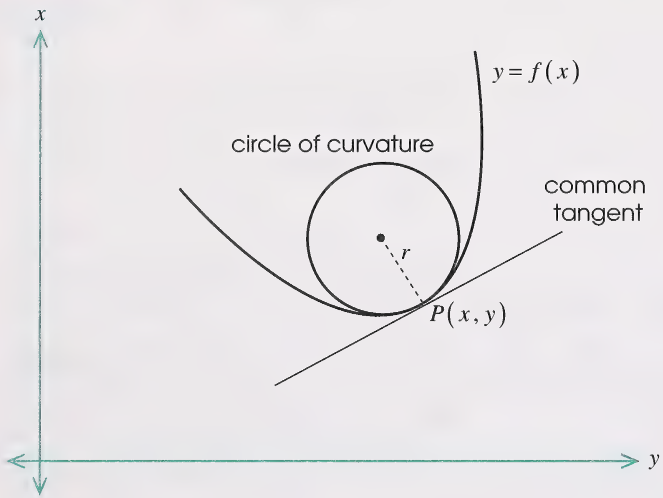
2. How would you describe the portion of the graph of $x^2 + y^2 = 4$ for which $f''(x) > 0$?

Check your answers by turning to the Appendix.

Enrichment

The curvature of a function $y = f(x)$ at a point $P(x, y)$ on its graph can be related to a circle that has the same tangent and the same second derivative at point P as the function itself. There is an infinte number of circles that can be drawn with the same tangent at P, but they will not have the same second derivative as function f at P. There is an infinite number of circles that have the same second derivative as function f at point P, but they will not have the same tangent.

There is only one circle that passes through P that has the same tangent and the same second derivative at P as the function itself. This circle is called the **circle of curvature**.



It can be shown that the radius of the circle of curvature is the value of the following expression at $P(x, y)$.

$$r = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}$$

Example

Find the radius of curvature for $f(x) = \frac{1}{2}x^2$ at $(0, 0)$. Draw a diagram.

Solution

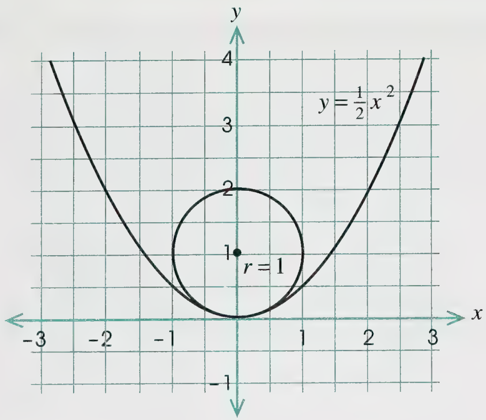
$$f'(x) = \frac{1}{2}(2x) \qquad f''(x) = 1$$
$$= x$$

Evaluate the derivatives at $(0, 0)$.

$$f'(0) = 0 \qquad f''(0) = 1$$

$$\therefore\ r = \frac{\left[1 + \left(f'(0)\right)^2\right]^{\frac{3}{2}}}{\left|f''(0)\right|}$$
$$= \frac{[1+0]^{\frac{3}{2}}}{1}$$
$$= 1$$

Since the tangent to $f(x) = \frac{1}{2}x^2$ at $(0, 0)$ is the x-axis, the centre of the circle of curvature is on the y-axis, at $(0, 1)$.



The circle and parabola are essentially the same at $(0, 0)$.

1. Find the radius of curvature.

a. $f(x)=x^3$ at $(1,1)$

b. $f(x)=\sqrt{x}$ at $(4,2)$

2. What is the radius of curvature of $x^2+y^2=4$ at $(0,-2)$?

Check your answers by turning to the Appendix.

Conclusion

In this section you used the second derivative to test for concavity and to locate maximum points, minimum points, and points of inflection. In particular, you should be able to do the following:

- explain how the sign of the second derivative indicates the concavity of a graph
- use the first and second derivatives to find maxima, minima, and inflection points to aid in graph sketching
- identify, from a graph, locations at which the first and second derivatives are zero
- illustrate, by examples, that a second derivative of zero is only one possible condition for an inflection point to occur

If you had difficulties with these concepts, review the relevant activities.

A long train, winding its way through mountain passes, twists and turns. Parts of the track trace out long *S*-shaped curves. As the train follows the track, the train first bends one way, and then the other. Did you ever consider that the point where the curvature changes is just like the of inflection points on the graph of a function?

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 5: Curve-Sketching Procedures

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Have you skied or walked through aspen forest in midwinter? Even at noon, the shadows of the trees stretch long across the snow. Dark shadows follow the contours, rising and falling, along the crests and valleys—the curvatures changing with the terrain. This photograph captures the complexity of those patterns on the snow.

How would you analyse and graph a complex curve? How would you approach the task? To sketch a function accurately, or to analyse it fully, you need a comprehensive strategy, making use of all relevant procedures at your disposal. You have mastered a number of curve-sketching techniques, both in this module and in Module 1. Recall that in Module 1 you investigated the combinations of functions, and how those combinations related to the resulting graphs. As well, you discussed transformations and symmetries of curves, and how to use those transformations and symmetries to simplify your work.

In this section you will use a systematic calculus procedure to sketch algebraic and trigonometric functions. You will make use of domain and range, intercepts, symmetries, asymptotes and discontinuities, transformations, intervals where the curve rises and falls, maximum and minimum points, concavity, and points of inflection.

Activity 1: Algebraic Functions

In the preceding sections, you were introduced to a number of procedures that assisted you in analysing the graphs of functions and relations. In this activity you will use your full repertoire of curve-sketching techniques, pulling together everything you have studied, to get as complete a picture as possible of what the graphs of various algebraic functions look like.

Graph-Sketching Prodcedures

Although not all of the following steps are used in sketching every curve, you may find these points a useful reminder of what features you should look for. The order of the steps may also vary from question to question.

Step 1: Determine the **domain** and **range** of the curve; the domain and range tell you where the curve is located in the plane.

Step 2: Find **intercepts**, if possible, to see if and where the graph crosses the axes.

Step 3: Locate **asymptotes**. They may tell you if there are any discontinuities. They will also help you to describe the curve as $|x|$ or $|y|$ increases.

Step 4: Decide if you can use any **symmetries** of the graph to your advantage; it may be possible, for example, to reflect the graph across the y-axis, eliminating lengthy calculations for points on the left side.

Step 5: Differentiate. Use f' to determine the intervals where the graph **increases** and where it **decreases**.

Step 6: Locate **extreme values**. Calculate y for values of x at points of transition between negative and positive slopes. These will yield maximum and minimum values. Investigate values of the function at endpoints of intervals for relative maxima or minima.

Step 7: Use f'' to test stationary points for extreme values. Find out where the curve is **concave upward** and **concave downward**.

Step 8: Locate the **inflection points**—the points of transition where concavity changes.

Step 9: Sketch a **smooth curve** through the points you determined, unless, of course, there are discontinuities.

Before applying these techniques in a few examples, watch the rest of the video titled *Derivatives and Graph Sketching* from the *Catch 31* series. This program is an excellent review of the outlined steps. Be sure to make notes and work through examples from the video as you view it. This video is available from the Learning Resources Distributing Centre.

Example 1

Use the outlined curve-sketching procedures to sketch the graph of $f(x) = -x^2(x-2)(x+2)$.

Solution

Domain: This is a polynomial function; its domain is the set of reals. The range cannot be determined until extreme values are located.

Intercepts: The x-intercepts occur when $f(x) = 0$. They are $x = 0$ and $x = \pm 2$. Since $f(0) = 0$, the y-intercept is 0.

Asymptotes: Polynomial functions do not have vertical, horizontal, or oblique asymptotes. There are no discontinuities.

Symmetries: none

Extreme values: Differentiate the function. Rather than applying the product rule, it is easier to multiply the factors together and then find the derivative.

$$\begin{aligned} f(x) &= -x^2(x-2)(x+2) \\ &= -x^2\left(x^2-4\right) \\ &= -x^4 + 4x^2 \end{aligned}$$

$$f'(x) = -4x^3 + 8x$$

Determine the stationary points.

$$\begin{aligned} f'(x) = -4x^3 + 8x &= 0 \\ -4x\left(x^2-2\right) &= 0 \end{aligned}$$

$$-4x = 0 \quad \text{or} \quad x^2 - 2 = 0$$
$$x = 0 \qquad\qquad x = \pm\sqrt{2}$$

$$\begin{aligned} f(0) &= (0)^4 + 4(0)^2 \\ &= 0 \end{aligned} \qquad \begin{aligned} f\left(\pm\sqrt{2}\right) &= -\left(\pm\sqrt{2}\right)^4 + 4\left(\pm\sqrt{2}\right)^2 \\ &= -4 + 4(2) \\ &= 4 \end{aligned}$$

Apply the Second Derivative Test to determine whether these values are relative minima or maxima.

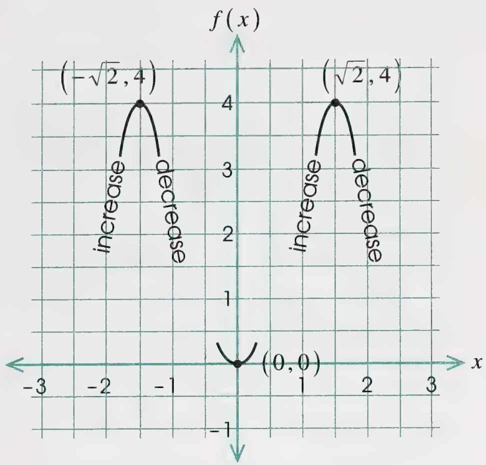
$$f''(x) = -12x^2 + 8$$

Since $f''(0) = 8 > 0$, then $(0, 0)$ is a minimum point.

$$\begin{aligned} f''\left(\pm\sqrt{2}\right) &= -12\left(\pm\sqrt{2}\right)^2 + 8 \\ &= -24 + 8 \\ &< 0 \end{aligned}$$

Therefore, $\left(\sqrt{2}, 4\right)$ and $\left(-\sqrt{2}, 4\right)$ are maximum points.

Intervals of increase or decrease: Plot the extreme points $\left(-\sqrt{2}, 4\right)$, $(0, 0)$, and $\left(\sqrt{2}, 4\right)$; where the function increases and decreases is automatic.



The function increases when $x < -\sqrt{2}$ or $0 < x < \sqrt{2}$. The function decreases when $-\sqrt{2} < x < 0$ or $x > \sqrt{2}$.

Concavity: If $f''(x) > 0$, then $-12x^2 + 8 > 0$

$$8 > 12x^2$$

$$12x^2 < 8$$

$$x^2 < \frac{2}{3}$$

$$-\frac{\sqrt{2}}{\sqrt{3}} < x < \frac{\sqrt{2}}{\sqrt{3}}$$

$$-\frac{\sqrt{6}}{3} < x < \frac{\sqrt{6}}{3}$$

The curve is concave upward in the interval $\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}\right)$.

If $f''(x) < 0$, $-12x^2 + 8 < 0$

$$x^2 > \frac{2}{3}$$

$$\therefore\ x < -\frac{\sqrt{6}}{3} \text{ or } x > \frac{\sqrt{6}}{3}$$

The curve is concave downward in the interval $\left(-\infty, -\frac{\sqrt{6}}{3}\right) \cup \left(\frac{\sqrt{6}}{3}, \infty\right)$.

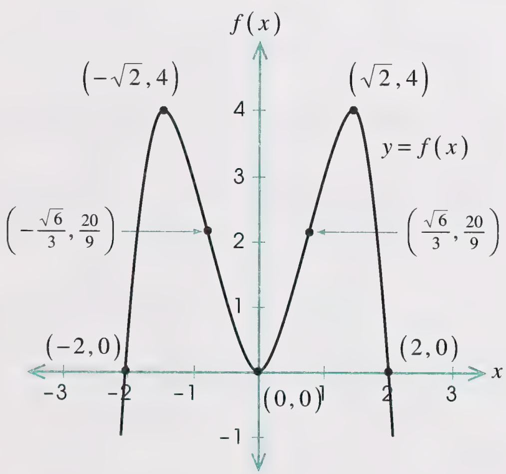
Points of inflection: The transition points where the curve changes concavity occur at $x = \pm\frac{\sqrt{6}}{3}$, where the value of the second derivative is zero.

Since $f\left(\pm\frac{\sqrt{6}}{3}\right)=-\left(\pm\frac{\sqrt{6}}{3}\right)^4+4\left(\pm\frac{\sqrt{6}}{3}\right)^2$

$$=-\frac{36}{81}+\frac{4(6)}{9}$$
$$=\frac{20}{9}$$

The inflection points are $\left(\frac{\sqrt{6}}{3},\frac{20}{9}\right)$ and $\left(-\frac{\sqrt{6}}{3},\frac{20}{9}\right)$.

Sketch: Plot the points you determined. Join the points with a smooth curve, taking both where the curve increases and decreases and the curve's concavity into account.



Example 2

Sketch the graph of $f(x)=\frac{x^2}{x-1}$.

Solution

Domain: The function is undefined at $x=1$. The domain is $(-\infty,1)\cup(1,\infty)$.

Intercepts: Since $f(0)=0$, the graph crosses both axes at the origin.

Asymptotes: The vertical asymptote is $x=1$; for that value of x the numerator of this rational function is non-zero, but its denominator is zero. The degree of the numerator is one more than the degree of the denominator; therefore, there is an oblique asymptote.

Using long division,
$$\begin{array}{r} x+1 \\ x-1\overline{)\,x^2+0x+0} \\ \underline{x^2-x} \\ x+0 \\ \underline{x-1} \\ 1 \end{array}$$

$$\therefore f(x)=x+1+\frac{1}{x-1}$$

The function approaches $x+1$ as $x\to\infty$. The oblique asymptote is $y=x+1$.

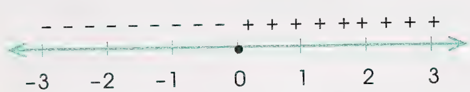
Symmetry: There is no symmetry.

Intervals of increase or decrease: Differentiate using the quotient rule.

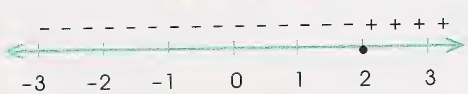
$$f'(x)=\frac{(x-1)\frac{d}{dx}\left(x^2\right)-x^2\frac{d}{dx}(x-1)}{(x-1)^2}$$
$$=\frac{(x-1)(2x)-x^2(1)}{(x-1)^2}$$
$$=\frac{x(x-2)}{(x-1)^2}$$

The numerator determines the sign of the derivative.

Sign of x



Sign of $(x-2)$



Sign of $x(x-2)$

The graph rises for $x<0$ and $x>2$.

The graph falls for $0<x<1$ and $1<x<2$. Recall that $x\neq 1$.

Extrema: From the discussion concerning where the function increases and where it decreases, it appears that turning points occur at $x=0$ and $x=1$. A look at the first derivative tells you that $f'(0)=0$ and $f'(2)=0$.

$$f(0)=\frac{(0)^2}{0-1} \quad \text{and} \quad f(2)=\frac{(2)^2}{2-1}$$
$$=0 \qquad\qquad\qquad =4$$

The point $(0,0)$ is a maximum since the graph rises to that point and then falls.

The point $(2,4)$ is a minimum since the graph falls to that point and then rises.

Concavity: Find the second derivative.

$$f''(x)=\frac{(x-1)^2\frac{d}{dx}\left(x^2-2x\right)-\left(x^2-2x\right)\frac{d}{dx}(x-1)^2}{\left[(x-1)^2\right]^2}$$

$$=\frac{(x-1)^2(2x-2)-\left(x^2-2x\right)(2)(x-1)(1)}{(x-1)^4}$$

$$=\frac{2(x-1)\left[(x-1)(x-1)-\left(x^2-2x\right)\right]}{(x-1)^4}$$

$$=\frac{2(x-1)\left[x^2-2x+1-x^2+2x\right]}{(x-1)^4}$$

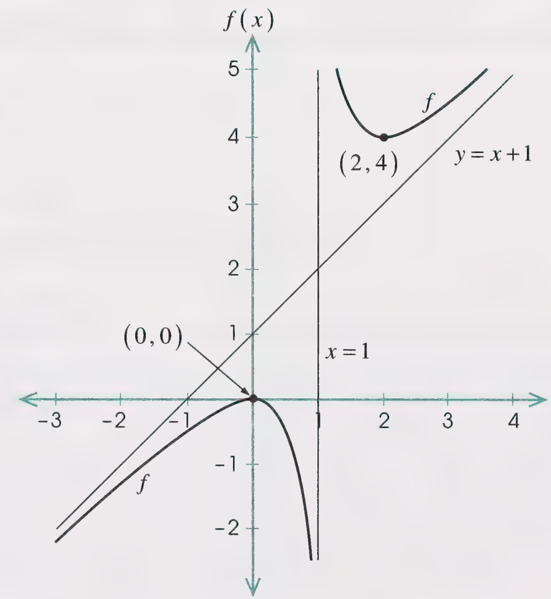
$$=\frac{2(x-1)}{(x-1)^4}$$

The sign of f'' depends on the numerator.

When $x>1$, $f''(x)>0$ and the curve is concave upward.

When $x<1$, $f''(x)<0$ and the curve is concave downward.

Sketch: Graph the asymptotes first; then graph all points obtained from the preceding discussion. Remember to draw the curve approaching the asymptotes.



Example 3

Sketch $f(x)=x^{\frac{1}{2}}(1-x)$.

Solution

Domain: Since $\sqrt{x}$ is defined only when $x\geq 0$, the domain is $[0,\infty)$.

Intercepts: When $f(x)=0$, $x^{\frac{1}{2}}(1-x)=0$.

$$x=0 \text{ or } 1-x=0$$
$$x=1$$

The graph crosses the x-axis at $(0,0)$ and $(1,0)$.

Since $f(0)=(0)$, the only y-intercept is 0.

Asymptotes: none

Symmetries: none

Intervals of increase and decrease: Differentiate using the product rule.

$$\begin{aligned} f'(x) &= x^{\frac{1}{2}}\frac{d}{dx}(1-x)+(1-x)\frac{d}{dx}\left(x^{\frac{1}{2}}\right) \\ &= x^{\frac{1}{2}}(-1)+(1-x)\left(\frac{1}{2}\right)x^{-\frac{1}{2}} \\ &= \frac{1}{2}x^{-\frac{1}{2}}[-2x+(1-x)] \\ &= \frac{1}{2}x^{-\frac{1}{2}}(1-3x) \end{aligned}$$

The first derivative is undefined at $(0,0)$.

When $f'(x)>0$, the factor $1-3x>0$

$$1<3x$$
$$x<\frac{1}{3}$$

The function increases on the interval $\left(0,\frac{1}{3}\right)$.

When $f'(x)<0$, $1-3x<0$

$$x>\frac{1}{3}$$

The function decreases on the interval $\left(\frac{1}{3},\infty\right)$.

Extrema: Since the graph rises for $0<x<\frac{1}{3}$ and falls for $x>\frac{1}{3}$, a maximum value must occur at $x=\frac{1}{3}$. This must be a stationary point since $f'\left(\frac{1}{3}\right)=0$.

$$\begin{aligned} f\left(\frac{1}{3}\right) &= \left(\frac{1}{3}\right)^{\frac{1}{2}}\left(1-\frac{1}{3}\right) \\ &= \frac{1}{\sqrt{3}}\left(\frac{2}{3}\right) \\ &= \frac{2}{3\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} \quad \text{(rationalize)} \\ &= \frac{2\sqrt{3}}{9} \end{aligned}$$

The point $\left(\frac{1}{3}, \frac{2\sqrt{3}}{9}\right)$ is a local maximum.

Don't forget to test the point at $x = 0$: the endpoint of the interval.

Since the curve rises for the interval $\left(0, \frac{1}{3}\right)$, $f(0) = 0$ must be a local minimum.

Concavity

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-3x)$$

$$f''(x) = \frac{1}{2}x^{-\frac{1}{2}}\frac{d}{dx}(1-3x) + (1-3x)\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$

$$f''(x) = \frac{1}{2}x^{-\frac{1}{2}}(-3) + (1-3x)\left(-\frac{1}{2}\right)\left(x^{-\frac{3}{2}}\right)$$

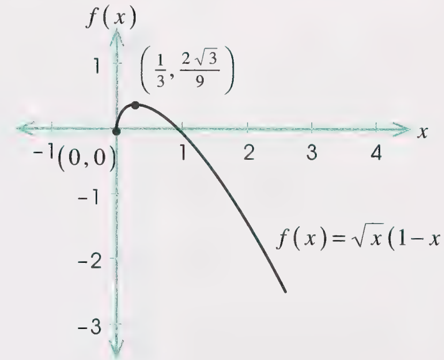
$$= -\frac{1}{2}x^{-\frac{3}{2}}(3x+1-3x)$$

$$= -\frac{1}{2}x^{-\frac{3}{2}}$$

$\therefore f''(x) < 0$ for all $x > 0$

The curve is concave downward for $x > 0$.

Sketch: The graph is as follows:



1. Use the steps outlined in this activity to sketch each function.

   a. $f(x) = \dfrac{x^2+1}{x^2-1}$ b. $f(x) = \left(x^2 - 4x\right)^{\frac{1}{2}}$

2. Sketch the graph of $y = 2\sqrt{x} + \dfrac{1}{x}$.

Check your answers by turning to the Appendix.

The steps outlined in this activity form a framework for sketching non-algebraic functions as well. In the next activity you will explore trigonometric functions using these procedures.

Activity 2: Trigonometric Functions

What does the seasonal change in daylight hours have to do with trigonometric functions and their graphs? Situations involving periodic or repetitive motion are often modelled using trigonometric functions: the motion of the planets, the rise and fall of the tides, the seasonal change in daylight hours, the motion of a pendulum. Being able to analyse those functions and their graphs using the tools of calculus is fundamental to a deeper understanding of the natural world.

In the examples which follow, you will use the same set of curve-sketching procedures outlined in Activity 1. Also, you will use what you studied about the graphs of trigonometric functions in Mathematics 30 and in Module 4.

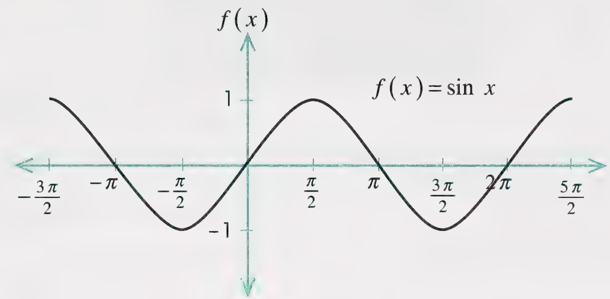
Example 1

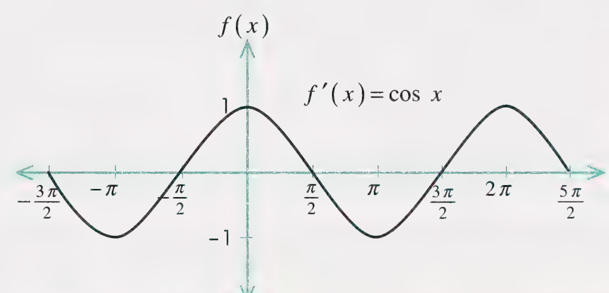
Sketch and describe the graphs of $y = \sin x$, and its first and second derivatives.

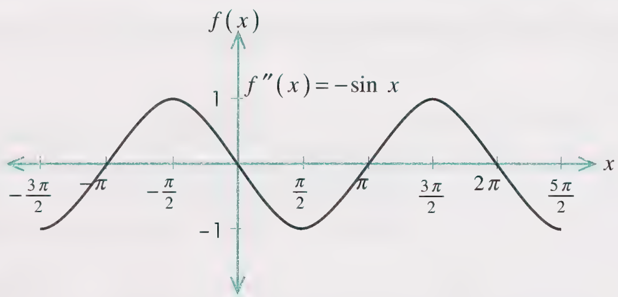
Solution

Differentiate.

$f(x) = \sin x \qquad f'(x) = \cos x \qquad f''(x) = -\sin x$







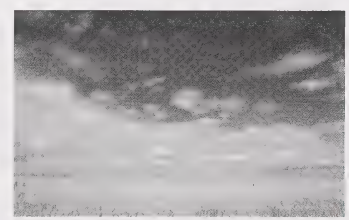
The maximum value of $\sin x$ is 1; this value occurs when $x = \frac{\pi}{2} + 2n\pi$, where $n \in I$. $f'(x) = \cos x$ is zero at these values, and $f''(x) < 0$. The minimum value of $\sin x$ is -1; this value occurs when $x = \frac{3\pi}{2} + 2n\pi$, where $n \in I$. $f'(x) = \cos x$ is also zero at these values, but $f''(x) > 0$. The critical values of x occur when $f'(x) = \cos x = 0$. As you can see, maximum and minimum values can be obtained or confirmed by applying the Second Derivative Test.

The graph of $f(x) = \sin x$ rises to the right when $f'(x) > 0$; that is, where the graph of $f'(x) = \cos x$ lies above the x-axis. For example, $f(x) = \sin x$ rises in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, where $\cos x > 0$. The graph of $f(x) = \sin x$ falls to the right when $f'(x) < 0$; that is, where the graph of $f'(x) = \cos x$ lies below the x-axis.

The graph of $f(x) = \sin x$ is concave upward when $f''(x) > 0$. For instance, $f''(x) = -\sin x$ is positive in the interval $(\pi, 2\pi)$. The graph of $f(x) = \sin x$ is concave downward when $f''(x) < 0$. For instance, $f''(x) = -\sin x$ is negative in the interval $(0, \pi)$. The point $(\pi, 0)$ is therefore a point of inflection on $f(x) = \sin x$. The entire set of points of inflection occurs at $x = n\pi$, where $f''(x) = -\sin x = 0$ and where $f'(x) = \cos x$ has extreme values.

1. What is the slope of $y = \sin x$ at $(0, 0)$? Are there any points on the graph of $y = \sin x$ where the slope is greater?

2. The number of daylight hours y on the xth day of the year, in southern Alberta, is approximated by $y = 4 \sin\left[\frac{2\pi(x-80)}{365}\right] + 12$. On what day of the year is the number of daylight hours increasing the fastest?



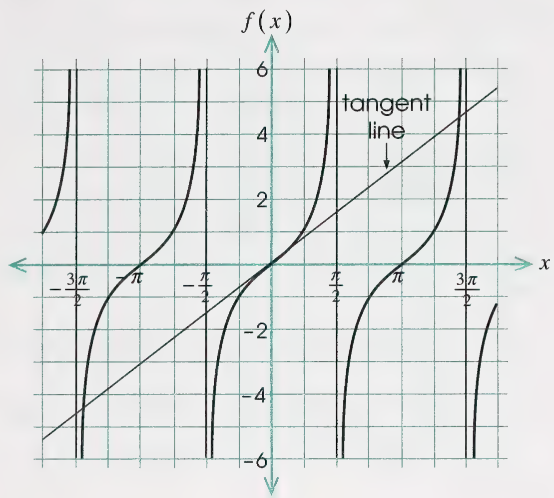
Check your answers by turning to the Appendix.

Example 2

Sketch the graph of $f(x) = \tan x$, showing the asymptotes. What is the equation of the tangent to the graph of $f(x) = \tan x$, at $x = 0$?

Solution

Since $\tan x = \frac{\sin x}{\cos x}$, asymptotes occur when $\cos x = 0$ or at $x = \frac{\pi}{2} + n\pi$, where $n \in I$.



Now, $f'(x) = \sec^2 x$.

Find the slope at $(0, 0)$.

$$
\begin{aligned}
f'(0) &= \sec^2(0) \\
&= (\sec 0)^2 \\
&= 1^2 \\
&= 1
\end{aligned}
$$

Using $y = mx + b$, since $m = 1$ and $b = 0$, the tangent line is $y = x$.

Next, you will look at questions which involve terms other than the trigonometric functions.

Example 3

Using the curve-sketching procedure, sketch and describe $f(x) = \sin x + x$, where $-\pi \le x \le 2\pi$.

Solution

Domain: The function is defined for $-\pi \le x \le 2\pi$.

Intercepts: Since $f(0) = 0$, the graph crosses the axes at the origin.

Asymptotes: none

Symmetries: If (x, y) is replaced by $(-x, -y)$, the function is unchanged.

$$-y = \sin(-x) + (-x)$$

Remember: $\sin(-x) = -\sin x$

$$-y = -\sin - x$$
$$y = \sin x + x$$

The graph is symmetric about the origin, from $x = -\pi$ to $x = \pi$.

Intervals of increase and decrease: Differentiate.

$$f'(x) = \cos x + 1$$

The function increases when $f'(x) > 0$.

$$\cos x + 1 > 0$$
$$\cos x > -1$$

This is true for all values of the function in the domain except $x = -\pi$ and $x = \pi$. At these points $\cos x = -1$. These are stationary points. Remember, stationary points occur when $f'(x) = 0$. However, the point at $x = -\pi$ is not strictly speaking a stationary point, as $f'(-\pi)$ is undefined; it is an endpoint!

Extrema: Since the graph never decreases, extreme values occur at the interval $[-\pi, 2\pi]$.

$$\begin{aligned} f(-\pi) &= \sin(-\pi) + (-\pi) \\ &= 0 - \pi \\ &= -\pi \quad \text{(a local minimum)} \end{aligned}$$

$$\begin{aligned} f(2\pi) &= \sin(2\pi) + (2\pi) \\ &= 2\pi \quad \text{(a local maximum)} \end{aligned}$$

Concavity: $f''(x) = -\sin x$

The graph is concave downward when $f''(x) < 0$.

$$-\sin x < 0$$
$$\sin x > 0$$
$$0 < x < \pi$$

The graph is concave upward when $f''(x) > 0$.

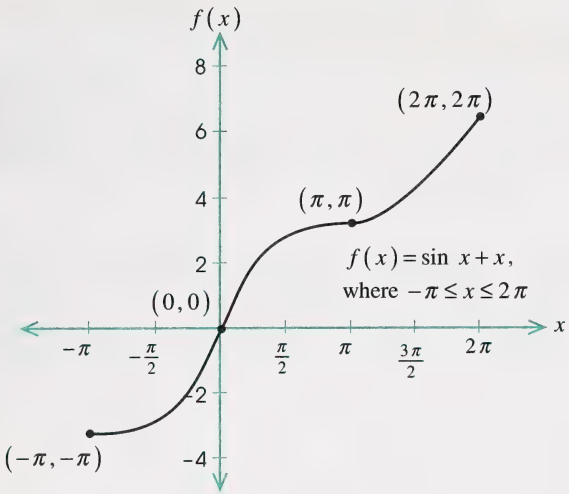
$$-\sin x > 0$$
$$\sin x < 0$$
$$-\pi < x < 0 \text{ and } \pi < x < 2\pi$$

Point of inflection: The points of inflection occur at $x = \pi$ and $x = 0$.

$$f(\pi) = \sin\ \pi + \pi$$
$$= 0 + \pi$$
$$= \pi$$

The points of inflection are $(0, 0)$ and (π, π).

Sketch: The graph is as follows:



3. Sketch $y = \sec\ x$, for $-\frac{3\pi}{2} < x < \frac{3\pi}{2}$.

Use a graphing calculator (or computer program) for question 4.

4. From its derivative, determine the critical values of the stationary points of $f(x) = \csc\ x$. Use a graphing calculator (or computer program) to show the relationship between the graphs of sine and cosecant.

5. Apply the Second Derivative Test to show that a local minimum occurs at $(0, 0)$ on the graph of $f(x) = x \sin\ x$.

6. Is the graph of $f(x) = \frac{\sin x}{x}$ increasing or decreasing at $x = -1$?

Check your answers by turning to the Appendix.

Did you know that trigonometric functions and other non-algebraic functions are termed **transcendental**?

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Primarily, this section has dealt with an algebraic approach to curve sketching. Of course, graphing calculators and graphing software packages (such as Zap-a-Graph™ and *Slope Explorer* on the *CAI-Math 31* program disk) allow you to graph quickly and accurately. What calculus gives you are the analytical tools to understand the behaviour of the graphs of particular functions and classes of functions. As well, calculus tells you where precisely in the plane to look for extrema, asymptotes, discontinuities, intervals of increase and decrease, concavity, and points of inflection. It is possible, when a function is graphed on a graphing calculator or computer program, that some of these features will not be displayed on the screen, as only a portion of the graph is displayed at any given time.

Nevertheless, computer graphing programs will reinforce what you have studied in this module, and will serve as a quick check on your algebra.

If you have access to a computer and the software package Zap-a-Graph™, use the software to work through those questions in the preceding activities that gave you trouble. In particular, use the "Derivative" and "Analyze" features under the Options menu. The "derivative" feature will display the derivative function superimposed on the graph of the original function. A quick comparison will tell you where the graph increases, decreases, or where stationary points are located. The "Analyze" feature will give you decimal numeral approximations for the location of the *y*-intercept, the extrema, and points of inflection. As you can see, this software is a useful tool for reinforcing what you have studied or for checking your algebra.

If you have access to the *CAI-Math 31* software package, the *Slope Explorer* will give you insight into the graphs of polynomial functions of degree 4 or less. The main screen displays the original function and the first derivative in factored form. Written text on the screen describes the relationship of the first derivative to the graph of the original function. As well, you can look at curvature, you can display the second derivative, and you can obtain an analysis of the concavity from point to point.

Graphing software is a useful aid to your understanding of calculus; its application depends on the imagination and knowledge of the user.

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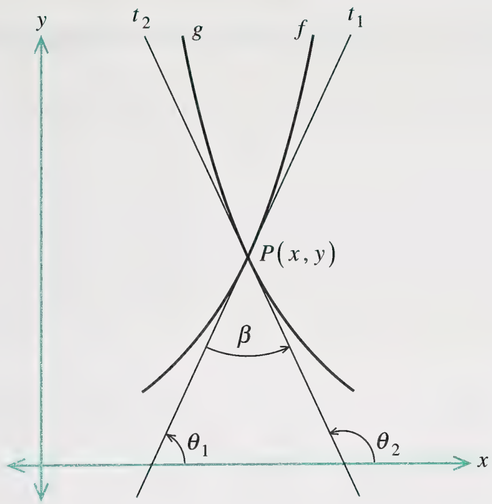
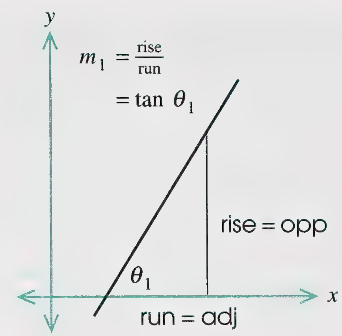
Enrichment

When two intersecting curves are sketched on the same graph, it may be of interest to determine the angle between the tangents to the curves at their point of intersection.





If functions f and g intersect at point $P(x, y)$, as shown, the angle between the curves is the angle β between their respective tangents t_1 and t_2, at P.

If the slopes of these tangents t_1 and t_2 are m_1 and m_2, then $\tan\,\beta$ can be determined as follows:

Note: $m_1 = \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{\text{opposite}}{\text{adjacent}}$

$= \tan\,\theta_1$

$m_2 = \tan\,\theta_2$

Since $\beta = \theta_2 - \theta_1$, $\tan\,\beta = \tan\,(\theta_2 - \theta_1)$

$$= \frac{\tan\,\theta_2 - \tan\,\theta_1}{1 + \tan\,\theta_2\ \tan\,\theta_1}$$

$$= \frac{m_2 - m_1}{1 + m_2 m_1}$$

This formula only works if neither tangent is vertical. Also, for perpendicular tangents, $m_1 m_2 = -1$, and the formula would not apply either!

Example

Find the angle between the graphs of $f(x)=x^2$ and $g(x)=\sqrt{x}$, at $x=1$.

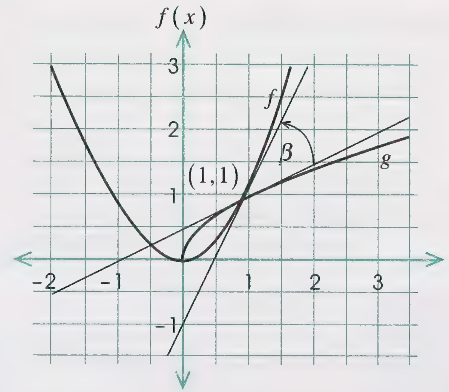
Solution

The graphs intersect at $(1,1)$.

Find the slopes of the tangent lines at these points.

$$g(x)=\sqrt{x} \qquad f(x)=x^2$$
$$g'(x)=\frac{1}{2}x^{-\frac{1}{2}} \qquad f'(x)=2x$$
$$g'(1)=\frac{1}{2} \qquad f'(1)=2$$
$$\therefore\ m_1=\frac{1}{2} \qquad \therefore\ m_2=2$$

$$\therefore\ \tan\beta=\frac{m_2-m_1}{1+m_2m_1}$$
$$\tan\beta=\frac{2-\frac{1}{2}}{1+2\left(\frac{1}{2}\right)}$$
$$\tan\beta=\frac{\frac{3}{2}}{1+1}$$
$$\tan\beta=\frac{3}{4}$$
$$\beta \doteq 36.87°$$



1. Find the angle between the graphs of $y=4x$ and $y=x^3$, at their point of intersection in the first quadrant.

2. Show that every curve of the family of curves $xy=a$, where a is a non-zero constant, is perpendicular to every curve of the family $x^2-y^2=b$, where b is a non-zero constant.

Check your answers by turning to the Appendix.

Conclusion

This was a summary section incorporating techniques you studied earlier in this module and in Module 1.

In this section you employed a systematic calculus procedure to sketch algebraic and trigonometric functions. You made use of the curve's domain and range, its intercepts, symmetries if any, asymptotes and discontinuities, transformations, intervals where the curve rises and falls, maximum and minimum points, concavity, and points of inflection.

Curves are like shadows on the snow. They are pictures of functions, just as shadows are only images. From the object itself you are able to predict what, generally, its shadow would look like. Having completed this section you should be able to use the tools of calculus to graph functions with considerably more precision.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Module Summary

In this module, differential calculus was presented as a powerful tool in determining maximum and minimum points on the graph of a function, intervals where the graph rises and falls, portions of the curve which are concave upward or concave downward, and points of inflection. These features of graphs, together with domain and range, and asymptotes and symmetry, were incorporated into a unified approach to curve sketching.

Remember, you will use these procedures for curve sketching not only in calculus, but also in other branches of mathematics and the sciences.

The artist in the photograph paints a representation of what he sees. He tries to recreate, for the people who view his painting, the beauty he experiences. In mathematics, the purpose for sketching graphs is to create a visual representation of a function or relation. Calculus provides you with the "brushes" and "paints" to become artists.

Final Module Assignment

Assignment Booklet

You are now ready to complete the final module assignment.

APPENDIX

Glossary

Suggested Answers

Glossary

absolute maximum: the value of a function which is not exceeded by any other value of that function

absolute minimum: the value of a function for which no other value of that function is less

asymptote: a line the graph of a function $y = f(x)$ approaches as x or y increases or decreases without bound

concavity: a measure of the curvature of a graph

critical values: the x-values of points on a curve where horizontal or vertical tangents occur

cusp: a point of sharp transition on a curve, where tangents on either side merge into a single tangent line

decreasing function: a function whose value decreases as the independent value increases

domain: the set of all x-values of the ordered pairs that comprise the relation

extrema: maximum or minimum values

First Derivative Test: determines whether a point is a relative minimum or maximum by exploring the slope of the curve on either side of that point

increasing function: a function whose value increases as the independent value increases

inflection point: a point on the graph where the curve changes from concave upward to concave downward, or vice versa

necessary condition: a mathematical statement which follows logically from a given statement

partitioning: dividing a set into disjoint subsets

range: the set of all y-values of the ordered pairs that comprise the relation

relation: a set of ordered pairs

relative (local) maxima: maximum values of a function over a specified interval

relative (local) minima: minimum values of a function over a specified interval

Second Derivative Test: determines whether a point is a relative minimum or relative maximum by investigating the convativity of the curve at that point

stationary points: points of contact on a curve with a horizontal tangent

sufficient condition: a mathematical statement from which another statement logically follows

turning points: points where a curve changes from rising to falling, or vice versa

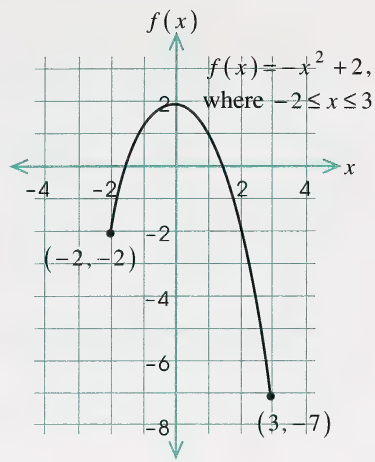
Suggested Answers

Section 1: Activity 1

1. The graph of f is a parabola which opens downward from its vertex $(0,2)$. The vertex is the maximum point on the graph. The minimum value of the function will occur at $x=3$.

$$\begin{aligned} f(3) &= -(3)^2+2 \\ &= -9+2 \\ &= -7 \end{aligned}$$

The domain is $[-2,3]$.
The range is $[-7,2]$.



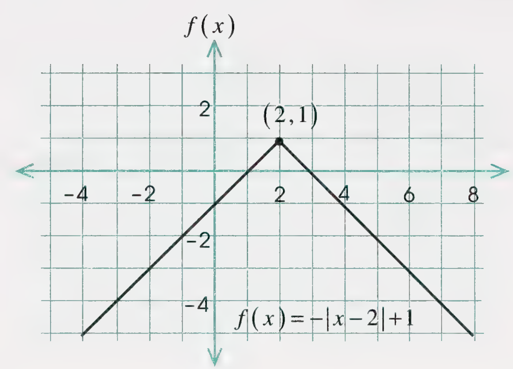
2. Read the domain and range directly from the graph. The domain is $[-2,3)$, and the range is $[-1,13]$.

3. a. Recall from Module 1 that this function is identical in shape to $y=-|x|$. The graph has been translated twice: horizontally 2 units and vertically 1 unit. The vertex is at $(2,1)$; the graph opens downward from that point.

Logically, since $-|x-2|$ is usually negative, except when $x=2$, the maximum value of the function must occur at that value of x.

$$\begin{aligned} f(2) &= -|2-2|+1 \\ &= -0+1 \\ &= 1 \end{aligned}$$

The domain is the set of reals, and the range is $y \le 1$ or $(-\infty,1]$.



b. Since division by zero is undefined, $x+3 \ne 0$ or $x \ne -3$.

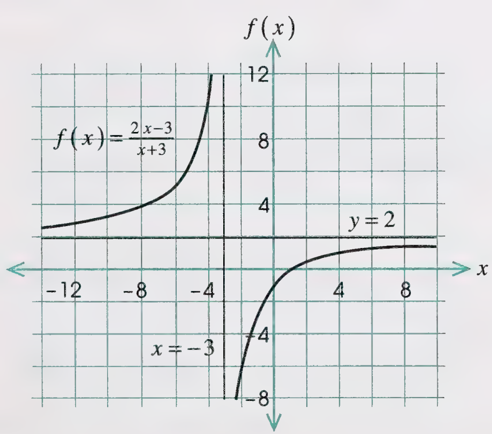
The domain is $(-\infty,-3)\cup(-3,\infty)$.

To find the restriction on y, rewrite the function with x as the subject.

$$y = \frac{2x-3}{x+3}$$
$$y(x+3) = 2x-3$$
$$xy+3y = 2x-3$$
$$xy-2x = -3y-3$$
$$x(y-2) = -3y-3$$
$$x = \frac{-3y-3}{y-2}$$

Since division by zero is undefined, $y-2 \neq 0$ or $y \neq 2$.

The range is $(-\infty, 2) \cup (2, \infty)$.



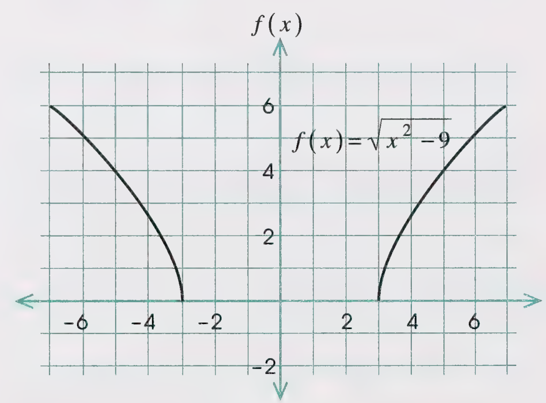
c. The square root of a negative number is undefined.

$$x^2 - 9 \geq 0$$
$$x^2 \geq 9$$
$$\therefore\ x \geq 3 \ \text{ or } \ x \leq -3$$

The domain is $[3, \infty) \cup (-\infty, -3]$.

Since $\sqrt{x^2 - 9}$ is a principal root, $y \geq 0$.

The range is $[0, \infty)$.

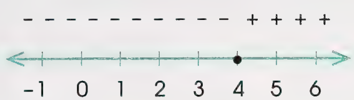


d. $-x^2+5x-4$ must be non-negative.

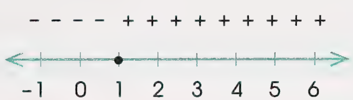
$$\begin{aligned} -x^2+5x-4 &\ge 0 \\ x^2-5x+4 &\le 0 \\ (x-4)(x-1) &\le 0 \end{aligned}$$

The product is zero when $x=1$ or $x=4$.

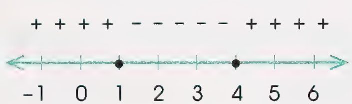
Sign of $(x-4)$



Sign of $(x-1)$

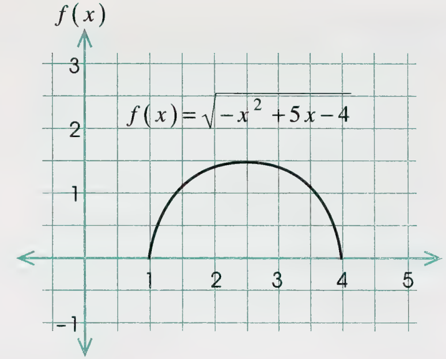


Sign of $(x-1)(x-4)$



The product is negative when $1<x<4$.

The domain is $[1,4]$.



e. Solve the equation for y.

$$\begin{aligned} x^2+4y^2 &= 4 \\ 4y^2 &= 4-x^2 \\ y^2 &= \frac{1}{4}\left(4-x^2\right) \\ y &= \pm\frac{1}{2}\left(4-x^2\right)^{\frac{1}{2}} \end{aligned}$$

$4-x^2$ must be non-negative.

$$\begin{aligned} 4-x^2 &\ge 0 \\ x^2 &\le 4 \\ \therefore\ -2 \le x &\le 2 \end{aligned}$$

The domain is $[-2, 2]$.

When $x = 0$, $\left(4 - x^2\right)^{\frac{1}{2}}$ is a maximum.

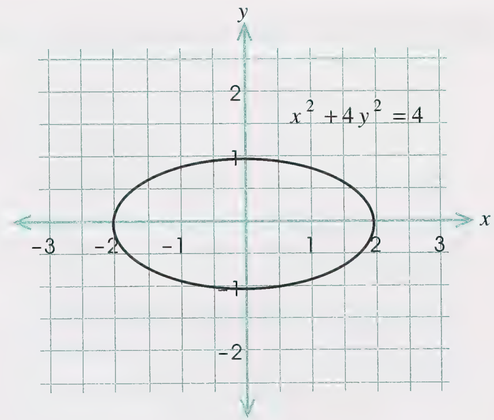
Therefore, $y = -\frac{1}{2}\left(4 - x^2\right)^{\frac{1}{2}}$ yields a minimum, and

$y = \frac{1}{2}\left(4 - x^2\right)^{\frac{1}{2}}$ yields a maximum.

$$\begin{aligned} \text{When } x = 0,\ y &= \pm\frac{1}{2}\left(4 - 0^2\right)^{\frac{1}{2}} \\ &= \pm\frac{1}{2}(4)^{\frac{1}{2}} \\ &= \pm\frac{1}{2}(2) \\ &= \pm 1 \end{aligned}$$

Thus, the range is $[-1, 1]$.

Notice that the domain and range could have been determined from the intercepts of the graph.



f. If $x > 0$, then $|x| = x$, and $f(x) = \frac{x}{x} = 1$.

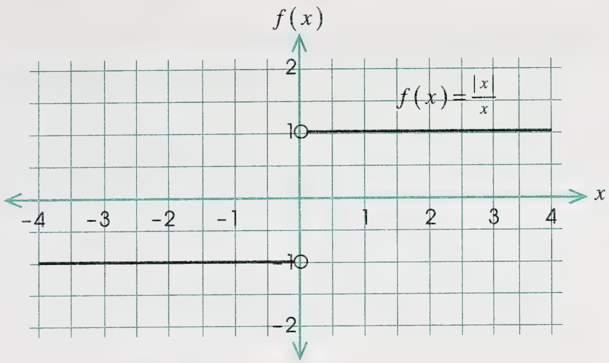
If $x < 0$, then $|x| = -x$, and $f(x) = -\frac{x}{x} = -1$.

$x \neq 0$, since $\frac{0}{0}$ is indeterminate.

The domain is $(-\infty, 0) \cup (0, \infty)$.

The range is $\{-1, 1\}$.

The graph is a step function with $f(0)$ undefined.



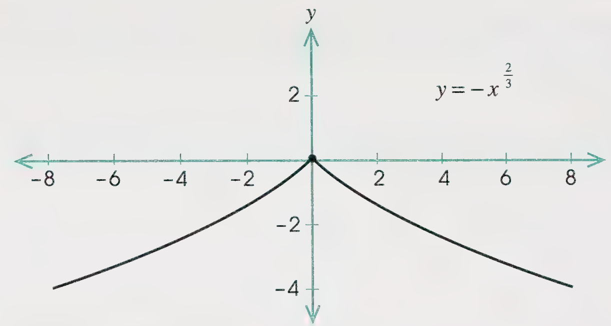
Section 1: Activity 2

1. **a.** The domain is $(-\infty,-2]\cup[3,\infty)$.
The range is the set of reals.

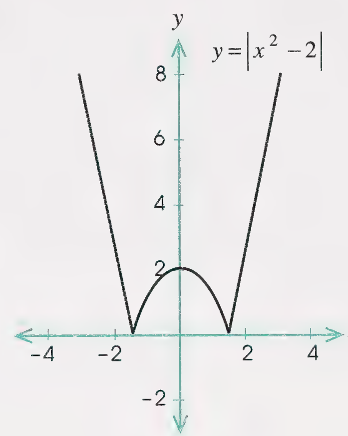
   b. The domain is $[-2,3]$.
The range is $[1,6]$.

2. Answers will vary. Some possible answers are as follows:
$y=2^x$, $y=\frac{1}{|x|}$, $y=\frac{1}{x^2}$.

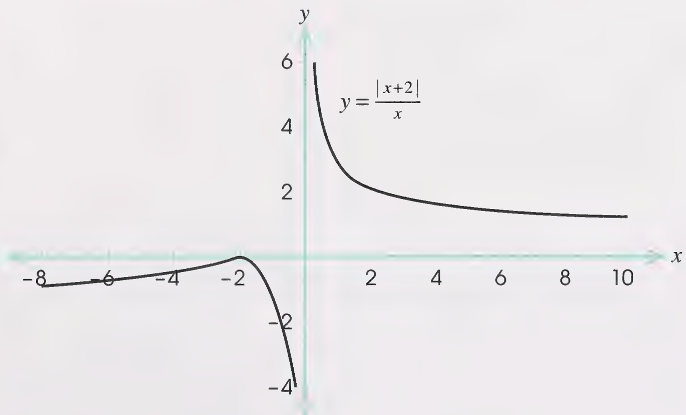
3. **a.** Because $-x^{\frac{2}{3}}$ is the same as $-\left(x^2\right)^{\frac{1}{3}}$, x may be any real value, and y is non-positive. Therefore, the domain is the set of reals and the range is $(-\infty,0]$.



   b. The domain is the set of reals. Since absolute value is non-negative, the range is $[0,\infty)$.



c. Since division by zero is undefined, the domain is $\{x \mid x \neq 0\}$. The range is $(-\infty, 0] \cup (1, \infty)$.



Section 1: Follow-up Activities

Extra Help

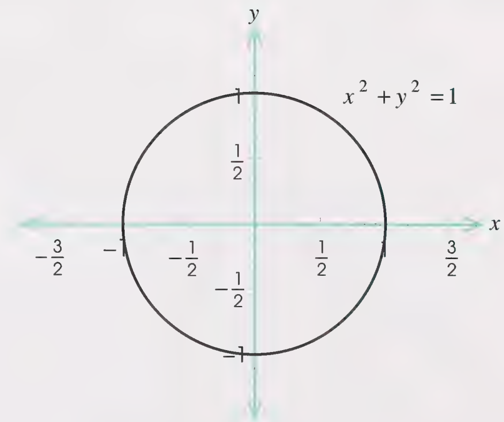
1. Find the x-intercepts.

Let $y = 0$.

$$\begin{aligned} x^2 + 0^2 &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

By symmetry, the y-intercepts are also ± 1.

Graph the circle.



The domain = range = $[-1, 1]$.

2. Find the x-intercepts.

Let $y = 0$.

$$\begin{aligned} 2x^2 + 5(0)^2 &= 40 \\ 2x^2 &= 40 \\ x^2 &= 20 \\ x &= \pm 2\sqrt{5} \end{aligned}$$

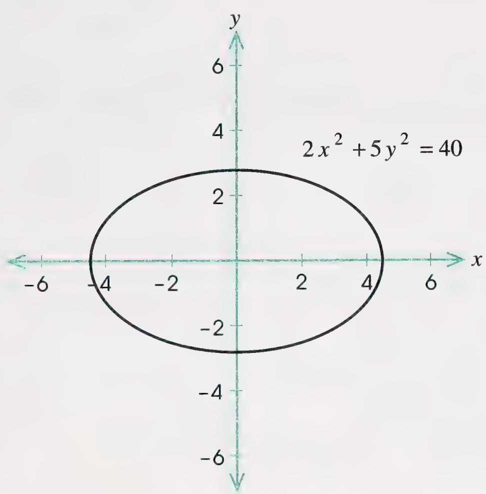
The graph crosses the x-axis at $(\pm 2\sqrt{5}, 0)$.

Find the y-intercepts.

Let $x = 0$.

$$2(0)^2 + 5y^2 = 40$$
$$5y^2 = 40$$
$$y^2 = 8$$
$$y = \pm 2\sqrt{2}$$

The graph crosses the y-axis at $\left(0, \pm 2\sqrt{2}\right)$.



The domain is $\left[-2\sqrt{5}, 2\sqrt{5}\right]$ and the range is $\left[-2\sqrt{2}, 2\sqrt{2}\right]$.

3. Find the x-intercepts.

Let $y = 0$.

$$x^2 - (0)^2 + 1 = 0$$
$$x^2 = -1$$

There are no x-intercepts.

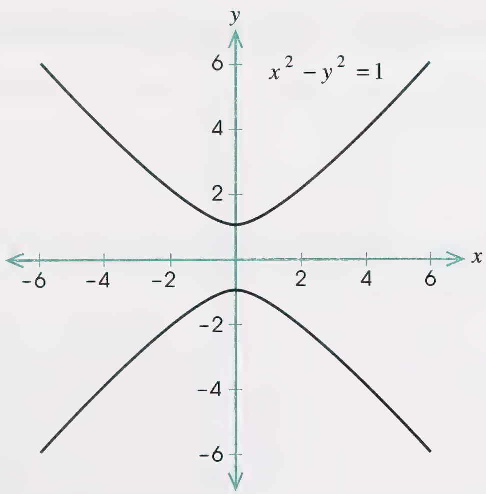
Find the y-intercepts.

Let $x = 0$.

$$(0)^2 - y^2 + 1 = 0$$
$$y^2 = 1$$
$$y = \pm 1$$

The graph crosses the y-axis at $(0, \pm 1)$.

This relation is a hyperbola which crosses the y-axis only.



The domain is the set of reals; the range is $(-\infty, -1] \cup [1, \infty)$.

4. Find the x-intercepts.

Let $y = 0$.

$$
\begin{aligned}
3x^2 - (0)^2 &= 1 \\
3x^2 &= 1 \\
x^2 &= \frac{1}{3} \\
x &= \pm\frac{1}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} \\
&= \pm\frac{\sqrt{3}}{3}
\end{aligned}
$$

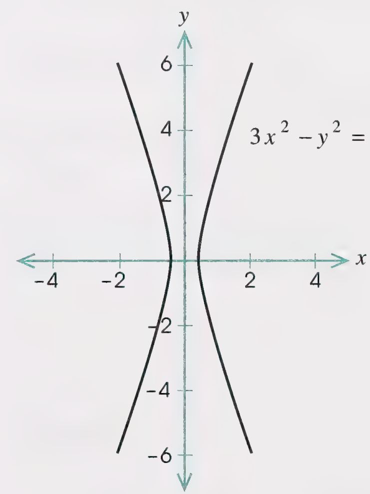
Find the y-intercepts.

Let $x = 0$.

$$
\begin{aligned}
3(0)^2 - y^2 &= 1 \\
y^2 &= -1
\end{aligned}
$$

The graph does not cross the y-axis.

This relation is a hyperbola which crosses the x-axis only.



The domain is $\left(-\infty, \frac{\sqrt{3}}{3}\right] \cup \left[\frac{\sqrt{3}}{3}, \infty\right)$; the range is the set of reals.

Enrichment

1. a. The relation $y^2 = x(x-4)$ is symmetric with respect to the x-axis since the equation remains unchanged when (x, y) is replaced by $(x, -y)$.

$$(-y)^2 = y^2 = x(x-4)$$

b. Since y^2 is non-negative, $x(x-4)$ must be 0 or positive.

$x(x-4)$ is zero at $x=0$ or $x=4$.
$x(x-4)>0$ when $x>4$ or $x<0$.

The domain is $(-\infty, 0]\cup[4, \infty)$.

The original relation may be written as follows:

$$\begin{aligned} y^2 &= x(x-4) \\ y^2 &= x^2 - 4x \\ x^2 - 4x - y^2 &= 0 \end{aligned}$$

Using the quadratic formula $x = -\frac{b \pm \sqrt{b^2-4ac}}{2a}$, solve for x, where $a=1$, $b=-4$, and $c=-y^2$.

$$\begin{aligned} x &= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-y^2)}}{2} \\ &= \frac{4 \pm \sqrt{16+4y^2}}{2} \end{aligned}$$

Since, $16+y^2$ is never negative, the range of the orginal relation is the set of reals.

c. Find the x-intercepts.

When $y=0$, $0^2 = x(x-4)$

$$\begin{aligned} x=0 \quad \text{or} \quad x-4&=0 \\ x&=4 \end{aligned}$$

The y-intercept is 0.

d.
$$\begin{aligned} y^2 &= x(x-4) \\ y^2 &= x^2 - 4x \\ 2yy' &= 2x-4 \\ y' &= \frac{2x-4}{2y} \\ &= \frac{x-2}{y} \end{aligned}$$

At $(4, 0)$ the slope is $y' = \frac{4-2}{0}$.

Since the slope is infinite, the tangent is vertical.

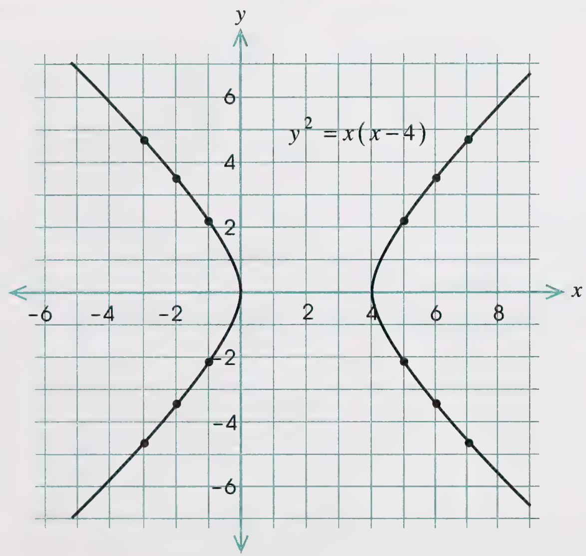
At $(0, 0)$, the slope is $y' = \frac{0-2}{0}$.

Again, the tangent is vertical.

e. Use the preceding information to graph the relation, and plot a few points as well.

x	5	6	7
y	$\pm\sqrt{5}$	$\pm\sqrt{12}$	$\pm\sqrt{21}$

x	-1	-2	-3
y	$\pm\sqrt{5}$	$\pm\sqrt{12}$	$\pm\sqrt{21}$



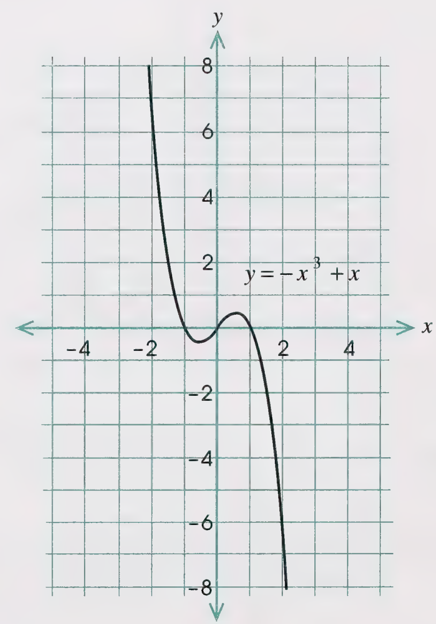
2. The graph is rotated counterclockwise through an angle of 90°.

Section 2: Activity 1

1. a. There is no horizontal asymptote.

$$\lim_{x\to+\infty}\left(-x^3+x\right)=-\infty \text{ and } \lim_{x\to-\infty}\left(-x^3+x\right)=+\infty$$

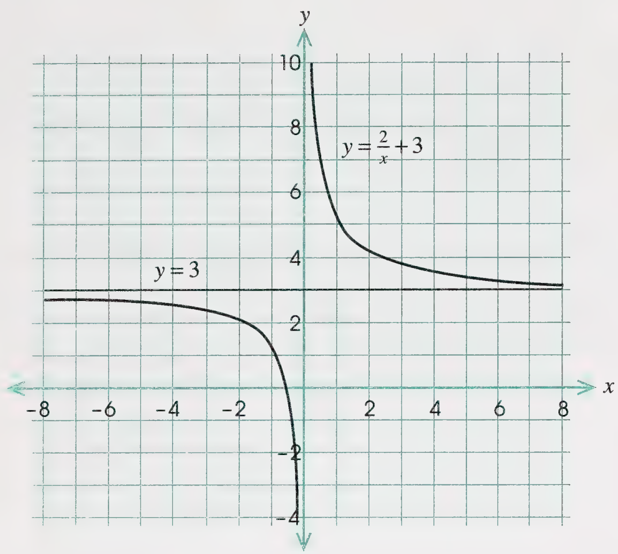
You should note that polynomial functions do not have asymptotes.



b. $\lim\limits_{x \to \infty} \left(\frac{2}{x} + 3 \right) = 0 + 3$

$= 3$

The horizontal asymptote is $y = 3$.

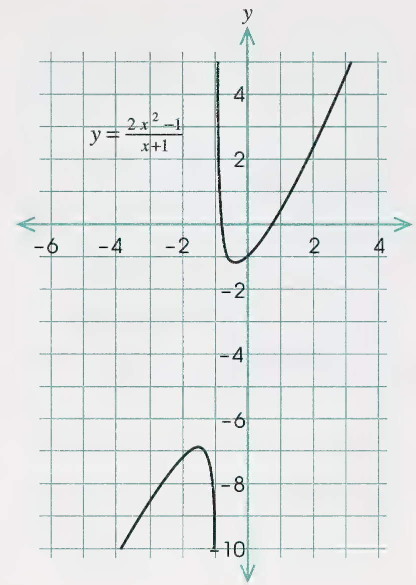


c. $\lim\limits_{x \to \infty} \frac{2x^2 - 1}{x + 1} = \lim\limits_{x \to \infty} \frac{x^2 \left(2 - \frac{1}{x^2} \right)}{x^2 \left(\frac{1}{x} + \frac{1}{x^2} \right)}$

$= \frac{2 - 0}{0 + 0}$

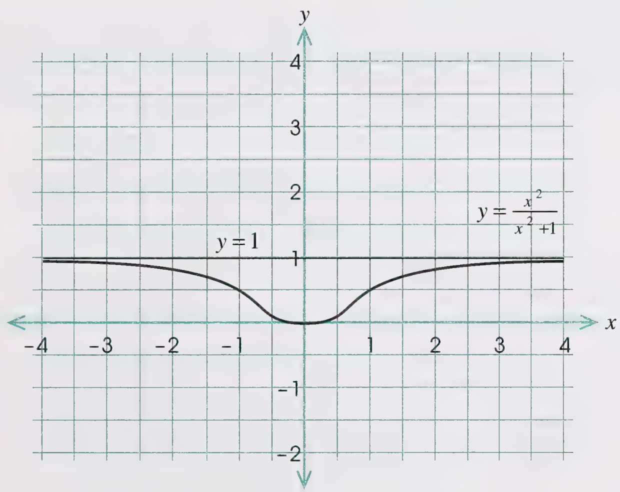
$=$ undefined

There is no horizontal asymptote.



2. a. $$\lim_{x\to\infty}\frac{x^2}{x^2+1}=\lim_{x\to\infty}\frac{x^2}{x^2\left(1+\frac{1}{x^2}\right)}$$
$$=\frac{1}{1+0}$$
$$=1$$

The horizontal asymptote is $y=1$.

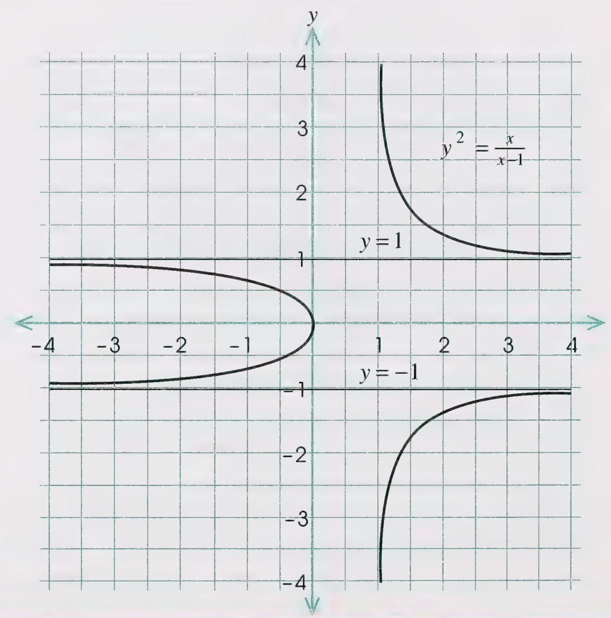


b. $y^2=\dfrac{x}{x-1}$

$$\lim_{x\to\infty}\frac{x}{x-1}=\lim_{x\to\infty}\frac{x}{x\left(x-\frac{1}{x}\right)}$$
$$=\frac{1}{1-0}$$
$$=1$$

As $x\to\infty$, $y^2\to 1$.

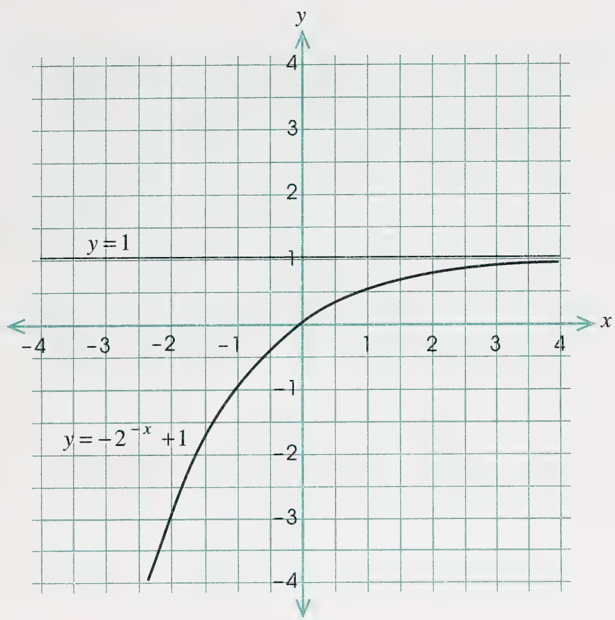
Thus, $y=1$ and $y=-1$ are the two asymptotes for this relation.



c. $\lim_{x \to +\infty} \left(-2^{-x}+1\right) = 0+1$

$= 1$

The horizontal asymptote is $y = 1$.

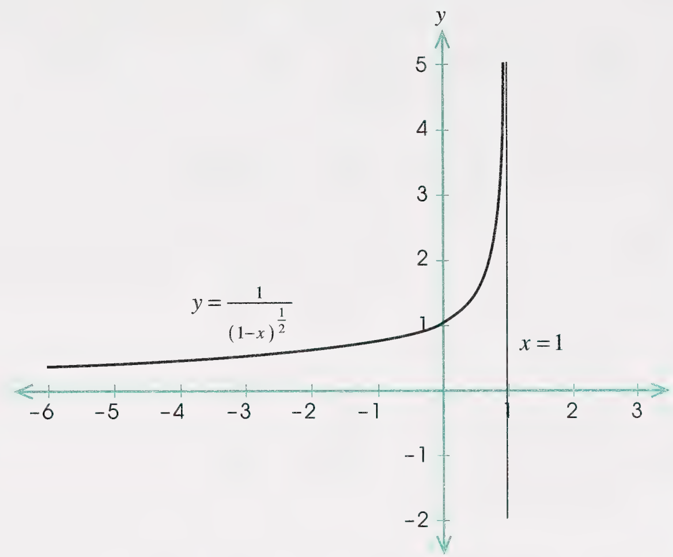


Section 2: Activity 2

1. a. $y = \dfrac{1}{(1-x)^{\frac{1}{2}}}$

Since $y \to 0$ as $x \to -\infty$, the horizontal asymptote is $y = 0$.

The vertical asymptote is $x = 1$.



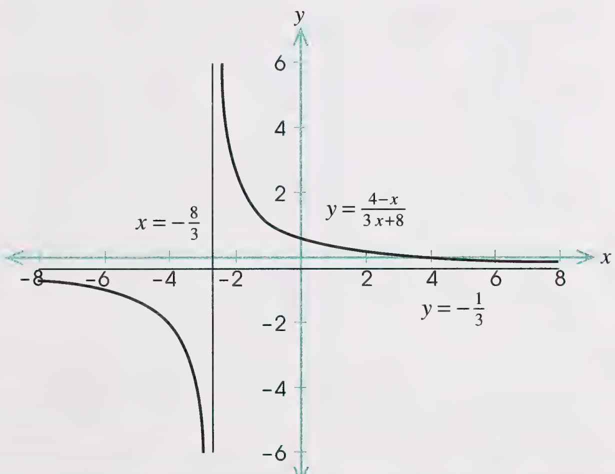
b. $y=\dfrac{4-x}{3x+8}$

The horizontal asymptote is $y=-\frac{1}{3}$.

$$\lim_{x\to\infty}\frac{4-x}{3x+8}=\lim_{x\to\infty}\frac{x\left(\frac{4}{x}-1\right)}{x\left(3+\frac{8}{x}\right)}$$
$$=\frac{0-1}{3+0}$$
$$=-\frac{1}{3}$$

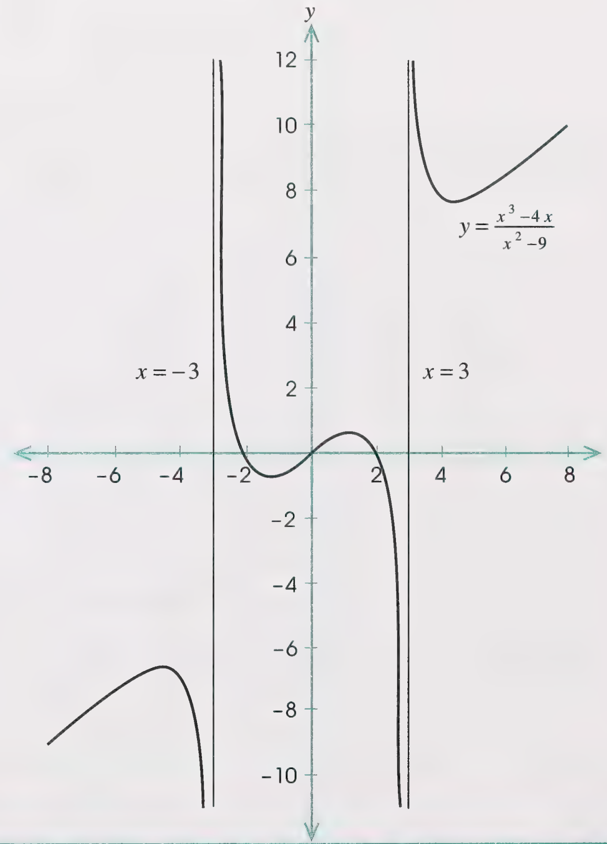
The vertical asymptote is $x=-\frac{8}{3}$.

If $3x+8=0$ then $x=-\frac{8}{3}$.



c. $y=\dfrac{x^3-4x}{x^2-9}$

Since the degree of the numerator is greater than that of the denominator, there are no horizontal asymptotes. The vertical asymptotes are $x=\pm 3$.



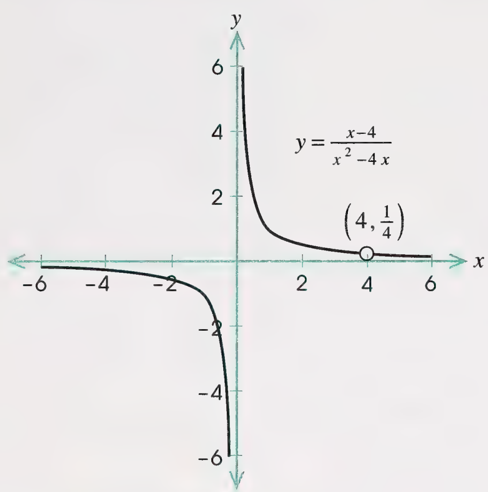
d. $y = \dfrac{x-4}{x^2-4x}$

$= \dfrac{x-4}{x(x-4)}$

$= \dfrac{1}{x}$, if $x \neq 4$

$\left(4, \frac{1}{4}\right)$ is a point of discontinuity.

The coordinate axes are the asymptotes.



2. $y^2 = \dfrac{x}{x-2}$

The vertical asymptote is $x = 2$.

$$\lim_{x\to\infty} \frac{x}{x-2} = \lim_{x\to\infty} \frac{x}{x\left(1-\frac{2}{x}\right)}$$

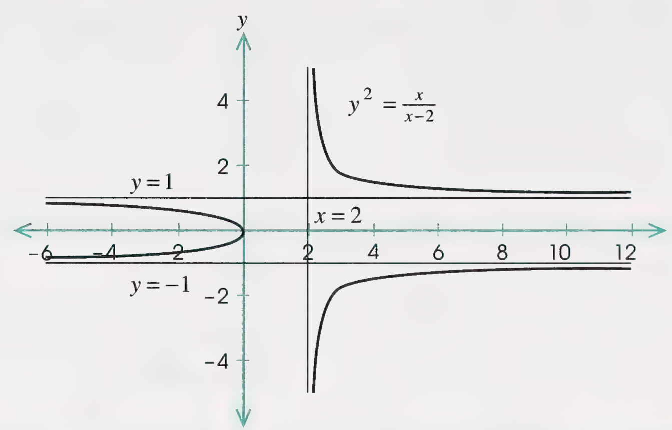
$$= \frac{1}{1-0}$$

$$= 1$$

$\therefore\ y^2 = 1$

$y = \pm 1$

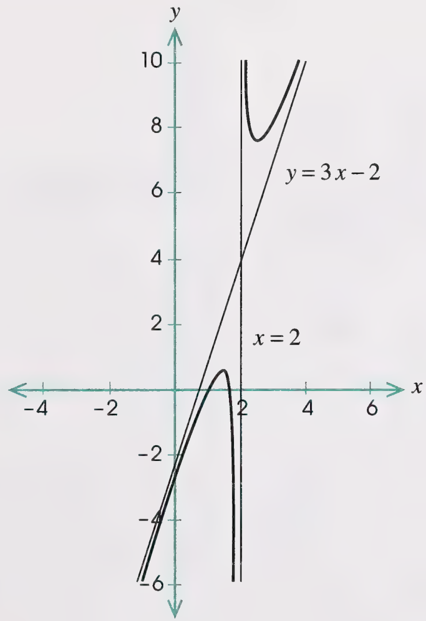
Thus, the horizontal asymptotes are $y = 1$ and $y = -1$.



Section 2: Activity 3

1. **a.** Vertical asymptote: $x = 2$
   Oblique asymptote: $3x - 2$

   There is no horizontal asymptote.



   b. $y = \dfrac{x^2 + x}{x - 2}$

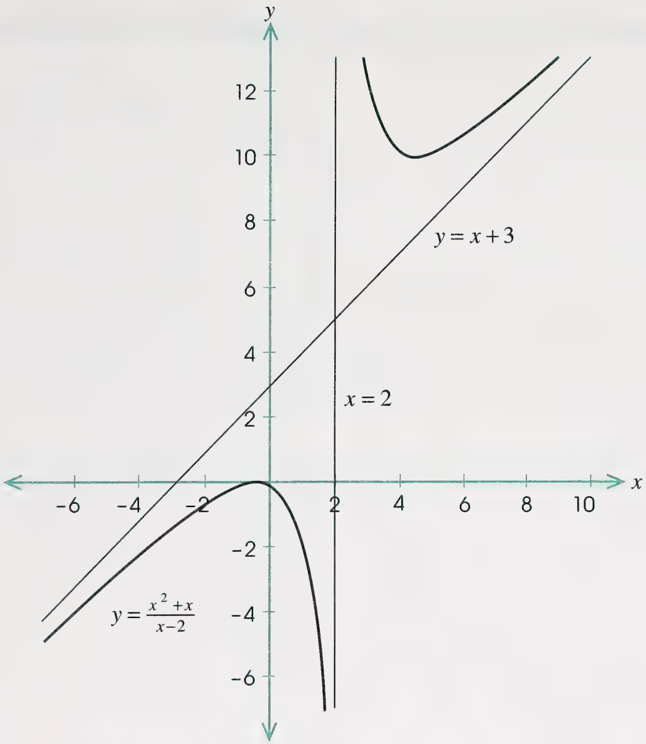
   Use long division.

$$\begin{array}{r}
x + 3 \\
x - 2\,\big)\overline{x^2 + \;\; x + 0} \\
\underline{x^2 - 2x \quad\;\;} \\
3x + 0 \\
\underline{3x - 6} \\
6
\end{array}$$

   Since $y = x + 3 + \frac{6}{x-2}$, the oblique asymptote is $y = x + 3$.

   The vertical asymptote is $x = 2$.

   There is no horizontal asymptote.



c. The vertical asymptotes are found by setting $x^2 - 4$ equal to 0.

$$x^2 - 4 = 0$$
$$x = \pm 2$$

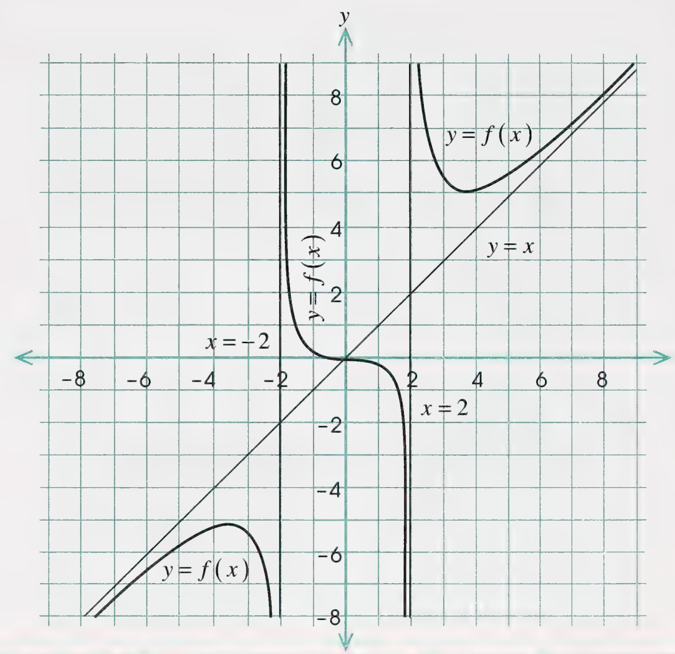
The vertical asymptotes are $x = 2$ and $x = -2$.

Find any oblique asymptotes.

$$x^2 - 4 \overline{\smash{)}\,x^3\,}\quad \text{quotient } x$$
$$\underline{x^3 - 4x}$$
$$4x$$

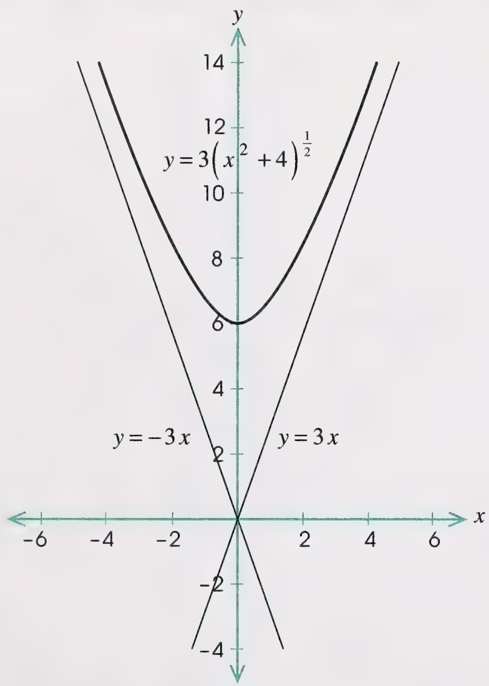
Now $y = x + \frac{4x}{x^2 - 4}$ and $\frac{4x}{x^2-4} \to 0$ as $x \to \infty$.

Therefore, $y = x$ is an oblique asymptote.



2. A rational function $y = \frac{P(x)}{Q(x)}$ cannot have both a slant asymptote and a horizontal asymptote. Horizontal asymptotes occur when the degree of $P(x)$ is less than or equal to the degree of $Q(x)$. Slant asymptotes occur when the degree of $P(x)$ is one more than the degree of $Q(x)$.

3. **a.** The slant asymptotes are $y = 3x$ and $y = -3x$.

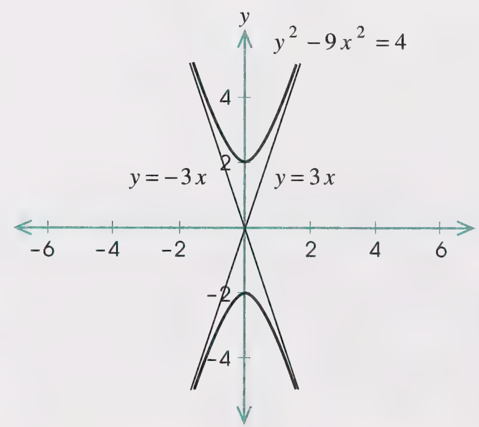


b. $y^2 = 9x^2 + 4$

$$y^2 = 9\left(x^2 + \frac{4}{9}\right)$$

$$y = \pm 3\left(x^2 + \frac{4}{9}\right)^{\frac{1}{2}}$$

The slant asymptotes are $y = 3x$ and $y = -3x$.



Section 2: Follow-up Activities

Extra Help

1. $y = \frac{x}{1-x}$

Find the horizontal asymptotes.

Method 1: Using Limits

$$\begin{aligned}\lim_{x\to 0}\frac{x}{1-x} &= \lim_{x\to 0}\frac{x}{x\left(\frac{1}{x}-1\right)}\\ &= \frac{1}{0-1}\\ &= -1\end{aligned}$$

Thus, the horizontal asymptote is $y=-1$

Method 2: Solving for x in Terms of y

$$\begin{aligned}y &= \frac{x}{1-x}\\ y(1-x) &= x\\ y-xy &= x\\ x+xy &= y\\ (1+y)x &= y\\ x &= \frac{y}{1+y}\end{aligned}$$

The denominator is 0 when $y=-1$.

Thus, the horizontal asymptote is $y=-1$.

Find the vertical asymptote.

Method 1: Finding the Zeros of the Denominator

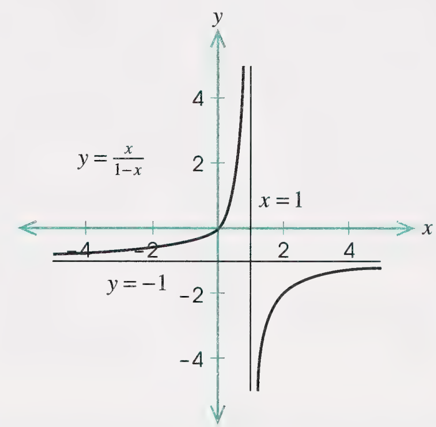
If $1-x=0$, then $x=1$.

Thus, the vertical asymptote is $x=1$.

Method 2: Using Limits

$$\begin{aligned}\lim_{y\to 0}\frac{y}{1+y} &= \lim_{y\to 0}\frac{y}{y\left(\frac{1}{y}+1\right)}\\ &= \frac{1}{0+1}\\ &= 1\end{aligned}$$

Thus, the vertical asymptote is $x=1$.



2. $y = \frac{1-2x}{x-2}$

Find the horizontal asymptotes.

Method 1: Using Limits

$$\begin{aligned}\lim_{x\to 0} \frac{1-2x}{x-2} &= \lim_{x\to 0} \frac{x\left(\frac{1}{x}-2\right)}{x\left(1-\frac{2}{x}\right)}\\ &= \frac{0-2}{1-0}\\ &= -2\end{aligned}$$

Thus, the horizontal asymptote is $y = -2$.

Method 2: Solving for x in Terms of y

$$\begin{aligned} y &= \frac{1-2x}{x-2}\\ y(x-2) &= 1-2x\\ xy-2y &= 1-2x\\ 2x+xy &= 1+2y\\ (2+y)x &= 1+2y\\ x &= \frac{1+2y}{2+y}\end{aligned}$$

The denominator is 0 when $y = -2$. Thus, the horizontal asymptote is $y = -2$.

Find the vertical asymptote.

Method 1: Finding the Zeros of the Denominator

If $x-2=0$, then $x=2$.

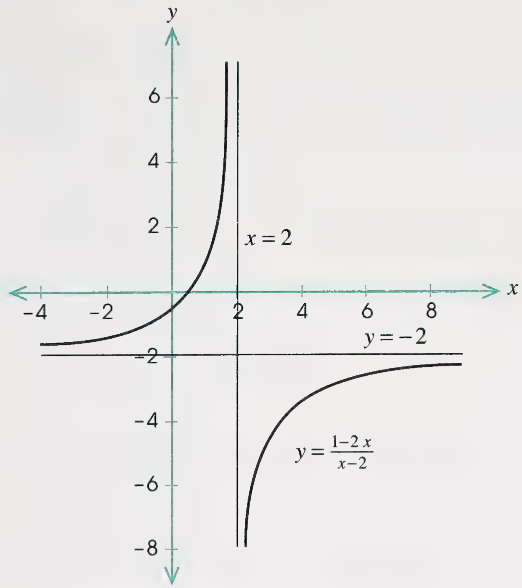
Thus, the vertical asymptote is $x = 2$.

Method 2: Using Limits

$$\begin{aligned}\lim_{y\to 0} \frac{1+2y}{2+y} &= \lim_{y\to 0} \frac{y\left(\frac{1}{y}+2\right)}{y\left(\frac{2}{y}+1\right)}\\ &= \frac{0+2}{0+1}\\ &= 2\end{aligned}$$

Thus, the vertical asymptote is $x = 2$.

Therefore, the graph looks as follows:



Enrichment

1. $$\lim_{x\to\infty}\left[\frac{x^3+1}{x^2}-x\right]=\lim_{x\to\infty}\left[\frac{x^3+1-x^3}{x^2}\right]$$
$$=\lim_{x\to\infty}\frac{1}{x^2}$$
$$=0$$

Therefore, $y = x$ is an oblique asymptote.

2. $$\lim_{x\to\infty}\left[\frac{x^3-x^2-1}{x^2+1}-(x-1)\right]$$
$$=\lim_{x\to\infty}\left[\frac{(x^3-x^2-1)-(x-1)(x^2+1)}{x^2+1}\right]$$
$$=\lim_{x\to\infty}\left[\frac{(x^3-x^2-1)-(x^3-x^2+x-1)}{x^2+1}\right]$$
$$=\lim_{x\to\infty}\left[\frac{x^3-x^2-1-x^3+x^2-x+1}{x^2+1}\right]$$
$$=\lim_{x\to\infty}\left[\frac{-x}{x^2+1}\right]$$
$$=\lim_{x\to\infty}\frac{x^2\left(-\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)}$$
$$=\frac{0}{1+0}$$
$$=0$$

Therefore, $y = x - 1$ is a slant asymptote.

Section 3: Activity 1

1. a. $y = x^2 + 2x - 3$

$$\frac{dy}{dx} = 2x + 2$$

Find the value of x for which the curve rises.

$$\begin{aligned} \text{When } \frac{dy}{dx} > 0,\ 2x + 2 &> 0 \\ 2x &> -2 \\ x &> -1 \end{aligned}$$

The graph rises when $x > -1$; thus, the function increases in the interval $(-1, \infty)$.

Find the values of x for which the curve falls.

$$\begin{aligned} \text{When } \frac{dy}{dx} < 0,\ 2x + 2 &< 0 \\ 2x &< -2 \\ x &< -1 \end{aligned}$$

The graph falls when $x < -1$; thus, the function decreases in the interval $(-\infty, -1)$.

Going from left to right along the curve, the graph first falls, then rises. A minimum point must occur. That transition point occurs at $x = -1$.

$$\begin{aligned} \text{When } x = -1,\ y &= (-1)^2 + 2(-1) - 3 \\ &= 1 - 2 - 3 \\ &= -4 \end{aligned}$$

The minimum point on the graph is $(-1, -4)$. Notice that the derivative is defined at $x = -1$, and $f'(-1) = 2(-1) + 2 = 0$. The slope at the minimum point is 0.

Find the x-intercepts by letting $y = 0$.

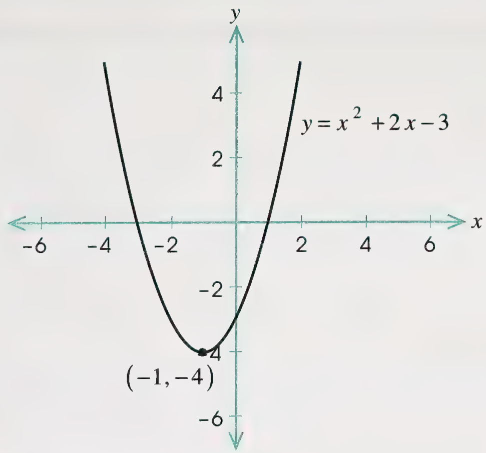
$$\begin{aligned} x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x + 3 = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = -3 \quad & \quad\quad x = 1 \end{aligned}$$

The graph crosses the x-axis at $(-3, 0)$ and $(1, 0)$.

Find the y-intercept by letting $x = 0$.

$$\begin{aligned} y &= 0^2 + 2(0) - 3 \\ &= -3 \end{aligned}$$

The graph crosses the y-axis at $(0, -3)$.



b. $y = -2x^2 + 4x - 2$

$$\frac{dy}{dx} = -4x + 4$$

Find the values of x for which the curve rises.

$$\begin{aligned} \text{When } \frac{dy}{dx} > 0,\ -4x + 4 &> 0 \\ -4x &> -4 \\ x &< 1 \end{aligned}$$

The graph rises when $x < 1$; thus, the function increases in the interval $(-\infty, 1)$.

Find the values of x for which the curve falls.

$$\begin{aligned} \text{When } \frac{dy}{dx} < 0,\ -4x + 4 &< 0 \\ -4x &< -4 \\ x &> 1 \end{aligned}$$

The graph falls when $x > 1$; thus, the function decreases in the interval $(1, \infty)$.

Going from left to right along the curve, the graph first rises, then falls. A maximum point must occur. That transition point occurs at $x = 1$.

$$\begin{aligned} \text{When } x = 1,\ y &= -2(1)^2 + 4(1) - 2 \\ &= -2 + 4 - 2 \\ &= 0 \end{aligned}$$

The maximum point on the graph is $(1, 0)$. Notice that the derivative is defined at $x = 1$, and $f'(-1) = -4(1) + 4 = 0$. The slope at the maximum point is 0.

Find the x-intercept by letting $y = 0$.

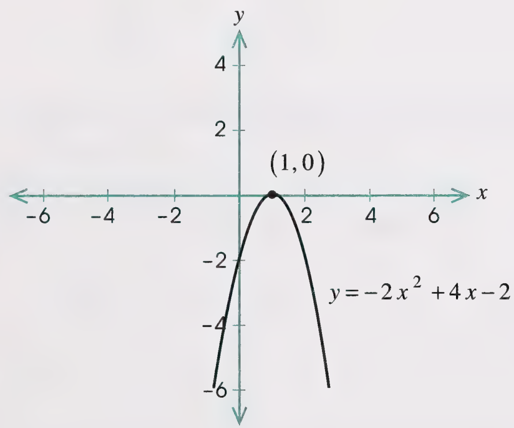
$$\begin{aligned} -2x^2 + 4x - 2 &= 0 \\ -2\left(x^2 - 2x + 1\right) &= 0 \\ -2(x-1)^2 &= 0 \\ x &= 1 \end{aligned}$$

The graph touches the x-axis at $(1, 0)$.

Find the y-intercept by letting $x = 0$.

$$\begin{aligned} y &= -2(0)^2 + 4(0) - 2 \\ &= -2 \end{aligned}$$

The graph crosses the y-axis at $(0, -2)$.



c. $y = x + \dfrac{9}{x}$

$$\begin{aligned} &= x + 9x^{-1} \\ \frac{dy}{dx} &= 1 - 9x^{-2} \end{aligned}$$

Find the values of x for which the curve rises.

When $\dfrac{dy}{dx} > 0$, $1 - 9x^{-2} > 0$

$$\begin{aligned} 1 &> 9x^{-2} \\ 1 &> \frac{9}{x^2} \\ x^2 &> 9 \end{aligned}$$

$\therefore\ x > 3$ or $x < -3$

The graph rises when $x > 3$ or $x < -3$; thus, the function increases in the interval $(-\infty, -3) \cup (3, \infty)$.

Find the values of x for which the curve falls.

When $\dfrac{dy}{dx} < 0$, $1 - 9x^{-2} < 0$

$$\begin{aligned} 1 &< 9x^{-2} \\ 1 &< \frac{9}{x^2} \\ x^2 &< 9 \end{aligned}$$

Now, the original function is undefined when $x = 0$.

$\therefore\ -3 < x < 0$ or $0 < x < 3$

The graph falls when $-3 < x < 0$ or $0 < x < 3$; thus, the function decreases in the interval $(-3, 0) \cup (0, 3)$.

At $x=-3$, the function changes from an increasing to a decreasing function; thus, a maximum point occurs at $x=-3$.

$$\text{When } x=-3,\ y=-3+\left(\frac{9}{-3}\right)$$
$$=-6$$

A maximum point occurs at $(-3,-6)$.

At $x=3$, the function changes from a decreasing to an increasing function; thus, a minimum point occurs at $x=3$.

$$\text{When } x=3,\ y=3+\frac{9}{3}$$
$$=6$$

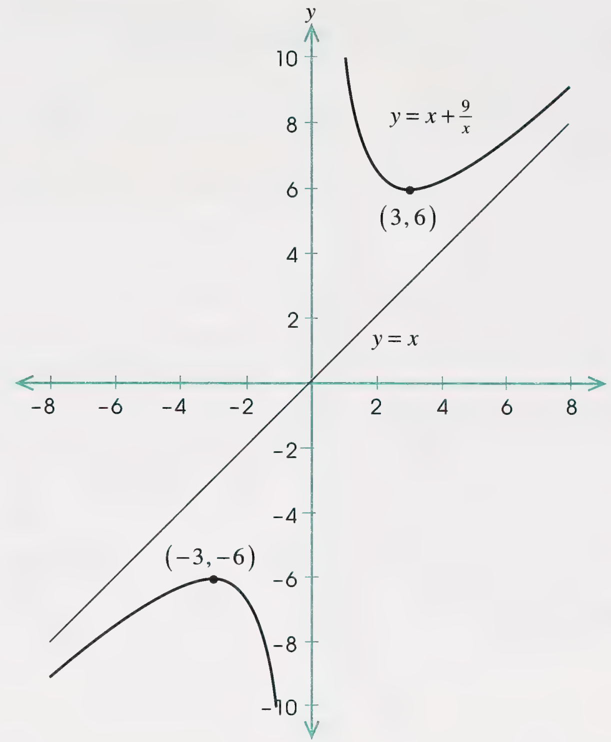
A minimum point occurs at $(3,6)$.

At both the minimum and maximum point, the derivative is defined and $\frac{dy}{dx}=0$. This can be shown as follows:

$$\begin{aligned}\text{When } f'(x)=0,\ 1-9x^{-2}&=0\\ 1&=9x^{-2}\\ 1&=\frac{9}{x^2}\\ x^2&=9\\ x&=\pm 3\end{aligned}$$

There is a vertical asymptote at $x=0$. As x increases or decreases without bound, the graph approaches $y=x$ (an oblique asymptote).

There are no x-intercepts or y-intercepts.



d. $y = \dfrac{x}{x+1}$

Differentiate using the quotient rule.

$$y' = \frac{vu' - uv'}{v^2}$$

$u = x$ and $v = x+1$

$u' = 1$ $\qquad v' = 1+0$

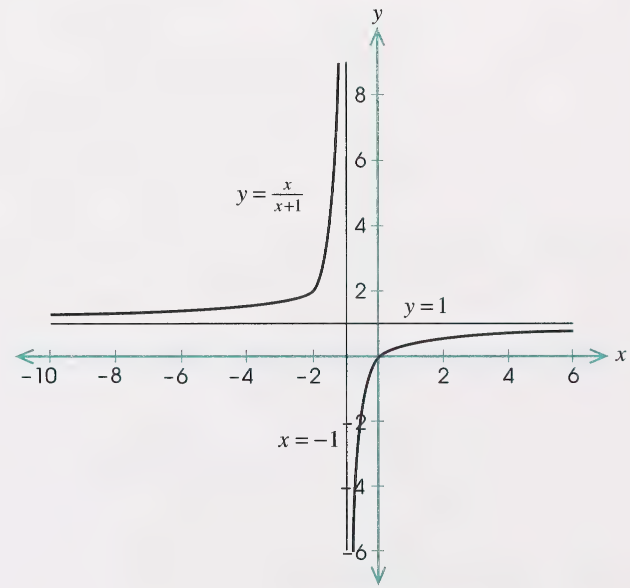
$$\therefore\ y' = \frac{(x+1)(1) - x(1+0)}{(x+1)^2}$$
$$= \frac{x+1-x}{(x+1)^2}$$
$$= \frac{1}{(x+1)^2}$$

Since the derivative is always positive, the function increases throughout its domain. There are no maxima or minima.

The vertical asymptote of $y = \frac{x}{x+1}$ is $x = -1$, because the function becomes infinte for that value of x. Therefore, the domain is $\{x | x \neq -1\}$.

Since $\lim\limits_{x \to \infty} \frac{x}{x+1} = 1$, the horizontal asymptote is $y = 1$.

Therefore, the range is $\{y | y \neq 1\}$.



2. At point A, the graph of the function is decreasing; the slope is negative.

At point B, the graph of the function has a maximum point; the slope is zero.

At point C, the graph of the function is decreasing; the slope is negative.

3. $f(x) = -x^3 + 12x - 1$

$f'(x) = -3x^2 + 12$

The function increases when $f'(x) > 0$.

$$\begin{aligned} -3x^2 + 12 &> 0 \\ 12 &> 3x^2 \\ x^2 &< 4 \\ -2 < x &< 2 \end{aligned}$$

Thus, the function increases when x lies in the interval $(-2, 2)$.

The function decreases when $f'(x) < 0$.

$$\begin{aligned} -3x^2 + 12 &< 0 \\ 12 &< 3x^2 \\ x^2 &> 4 \\ x < -2 \text{ or } x &> 2 \end{aligned}$$

Thus, the function decreases when x lies in the interval $(-\infty, -2) \cup (2, \infty)$.

This function will have a maximum or minimum when $f'(x) = 0$.

$$\begin{aligned} -3x^2 + 12 &= 0 \\ 12 &= 3x^2 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

When $x = 2$, $y = -2^3 + 12(2) - 1$
$= 15$

The point $(2, 15)$ is a maximum because the function is increasing immediately to the left of $x = 2$, and is decreasing immediately to the right of $x = 2$.

When $x = -2$, $y = -(-2)^3 + 12(-2) - 1$
$= -17$

The point $(-2, -17)$ is a minimum because the function is decreasing immediately to the left of $x = -2$, and is increasing immediately to the right of $x = -2$.

Section 3: Activity 2

1. The relative minima are B and D.
The relative maxima are A, C, and E.
There is no absolute minimum because $f(x) \to -\infty$ as $x \to +\infty$.
The absolute maximum is E.
The turning points are B, C, D, and E.
The stationary points are B, D, E, and F.

2. The critical values of x are those values for which $f'(x) = 0$ (horizontal tangent) or for which $f'(x)$ is infinite (vertical tangents). Since $f'(x)$ is not infinite for finite values of x, the only critical values are the solutions to $x^2 - 3x + 2 = 0$.

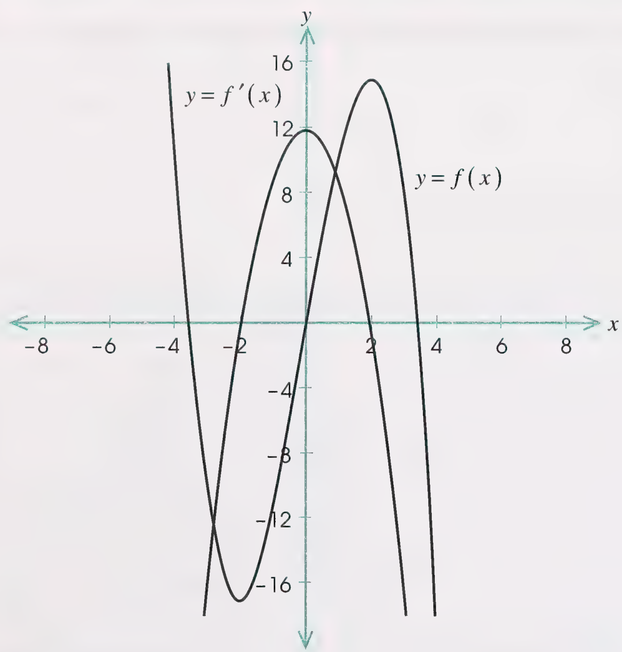
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 1 \qquad\qquad x = 2$$

Therefore, the critical values are $x = 1$ and $x = 2$.

3. $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$$= \frac{1}{3x^{\frac{2}{3}}}$$

$f'(x)$ is never 0; however, it is infinte when $x = 0$. Thus, $x = 0$ is a critical value.



4. The function increases for the following values of x: $-2 < x < 2$ or $x > 4$. These are obtained from $f'(x) > 0$. The function decreases for the following values of x: $x < -2$ or $2 < x < 4$. These are obtained from $f'(x) < 0$.

A minimum occurs at $x = -2$ and at $x = 4$, because in each case, the graph of the original function falls, then rises.

A maximum occurs at $x = 2$, because the graph of the original function rises, then falls.

4. Check for stationary points.

$$f(x)=\frac{2}{3}x^3-x^2-24x+2$$

$$\begin{aligned}f'(x)&=\frac{2}{3}\left(3x^2\right)-2x-24\\&=2x^2-2x-24\end{aligned}$$

To find the critical values, let $f'(x)=0$.

$$\begin{aligned}2x^2-2x-24&=0\\2\left(x^2-x-12\right)&=0\\2(x-4)(x+3)&=0\end{aligned}$$

$$x-4=0 \quad \text{or} \quad x+3=0$$
$$x=4 \qquad\qquad x=-3$$

Therefore, the critical values of *x* are $x=4$ and $x=-3$.

Evaluate the function for these values of *x*, and check to see if they are maximum values, minimum values, or neither.

$$\begin{aligned}f(4)&=\frac{2}{3}(4)^3-(4)^2-24(4)+2\\&=-\frac{202}{3}\end{aligned}$$

$$\begin{aligned}f(-3)&=\frac{2}{3}(-3)^3-(-3)^2-24(-3)+2\\&=47\end{aligned}$$

Begin by checking $f(4)=-\frac{202}{3}$.

Check values of the function in the neighbourhood, at points on **both** sides of $\left(4,-\frac{202}{3}\right)$.

$$\begin{aligned}f(5)&=\frac{2}{3}(5)^3-(5)^2-24(5)+2\\&=\frac{250}{3}-25-120+2\\&=-\frac{179}{3}\end{aligned}$$

$$\begin{aligned}f(3)&=\frac{2}{3}(3)^3-(3)^2-24(3)+2\\&=18-9-72+2\\&=-61\end{aligned}$$

Since both function values are larger than $f(4)=-\frac{202}{3}$, that value must be a relative minimum. This can be confirmed by looking at the slope of the curve on either side.

$$\begin{aligned}f'(3)&=2(3)^2-2(3)-24\\&=18-6-24\\&<0\end{aligned}$$

On the left, the graph is falling.

$$\begin{aligned} f'(5) &= 2(5)^2 - 2(5) - 24 \\ &= 50 - 10 - 24 \\ &> 0 \end{aligned}$$

On the right, the graph is rising.

Because the graph falls to point $\left(4, -\frac{202}{3}\right)$ and then rises, the point is a local minimum.

Next check $f(-3) = 47$.

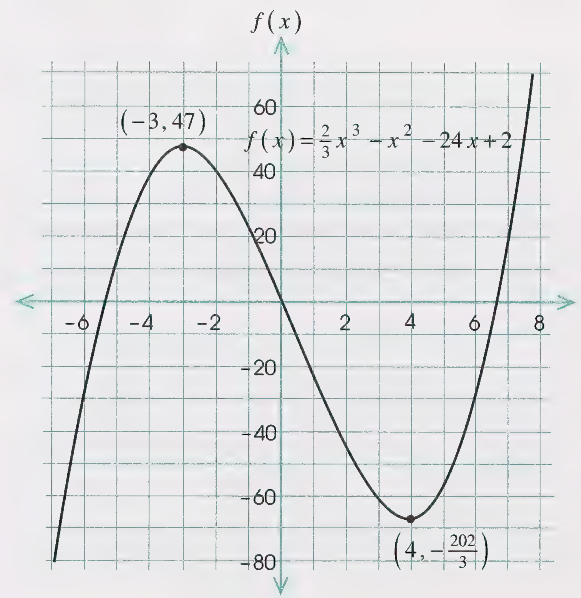
Check values of the function in the neighbourhood, at points on both sides of $(-3, 47)$.

$$\begin{aligned} f(-4) &= \frac{2}{3}(-4)^3 - (-4)^2 - 24(-4) + 2 \\ &= -\frac{128}{3} - 16 + 96 + 2 \\ &= \frac{118}{3} \end{aligned}$$

$$\begin{aligned} f(-2) &= \frac{2}{3}(-2)^3 - (-2)^2 - 24(-2) + 2 \\ &= -\frac{16}{3} - 4 + 48 + 2 \\ &= \frac{122}{3} \end{aligned}$$

Since both function values are smaller than $f(-3) = 47$, that value must be a relative maximum.

The graph looks as follows:



There are no absolute maximum or minimum values.

5. **a.** $f(x)=(x-2)^4+1$

$f'(x)=4(x-2)^3$

Stationary points occur when $f'(x)=0$.

If $f'(x)=0$, then $4(x-2)^3=0$.

Therefore, the critical value of x is 2.

$$\begin{aligned} f(2) &= (2-2)^4+1 \\ &= 0+1 \\ &= 1 \end{aligned}$$

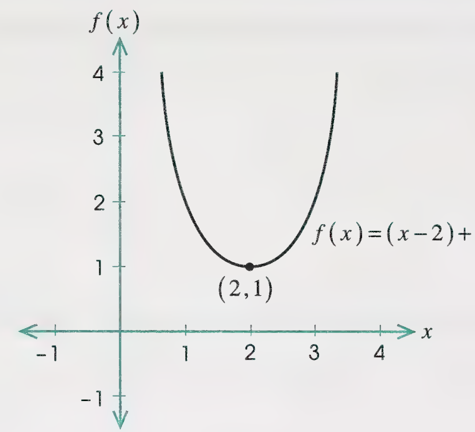
Determine whether $(2,1)$ is a maximum point, minimum point, or neither.

Test the slope on either side of $x=2$.

If $x<2$, the derivative $f'(x)=4(x-2)^3$ is negative. Thus, the curve falls.

If $x>2$, the derivative $f'(x)=4(x-2)^3$ is positive. Thus, the curve rises.

Therefore, at $x=2$, the functional value $f(2)=1$ is a relative minimum.



$f(2)=1$ is an absolute minimum.

b. $f(x)=\dfrac{x^2}{1-x}$

Differentiate using the quotient rule.

$$f'(x)=\frac{vu'-uv'}{v^2}$$

$u=x^2$ and $v=1-x$

$u'=2x$ $\quad$ $v'=-1$

$$\begin{aligned}\therefore f'(x) &= \frac{(1-x)(2x)-x^2(-1)}{(1-x)^2}\\ &= \frac{2x-2x^2+x^2}{(1-x)^2}\\ &= \frac{2x-x^2}{(1-x)^2}\end{aligned}$$

Stationary points occur when $f'(x)=0$.

If $f'(x)=0$, then
$$\begin{aligned}\frac{2x-x^2}{(1-x)^2} &= 0\\ 2x-x^2 &= 0\\ x(2-x) &= 0\end{aligned}$$

Thus, the critical values are $x=0$ and $x=2$.

Determine the points on the graph corresponding to these values.

$$\begin{aligned}f(0) &= \frac{0^2}{1-0}\\ &= 0\end{aligned}$$

Test the slope of the curve on either side of $x=0$.

$$\begin{aligned}f'(-0.1) &= \frac{2(-0.1)-(-0.1)^2}{[1-(-0.1)]^2}\\ &= \frac{-0.2-0.01}{1.1^2}\\ &< 0\end{aligned}$$

Therefore, the curve falls to the left of $(0,0)$.

$$\begin{aligned}f'(0.1) &= \frac{2(0.1)-(0.1)^2}{[1-(-0.1)]^2}\\ &= \frac{0.2-0.01}{1.1^2}\\ &> 0\end{aligned}$$

Therefore, the curve rises to the right of $(0,0)$.

Thus, the point $(0,0)$ is a minimum.

Next, find $f(2)$ and determine the nature of the resulting point.

$$\begin{aligned}f(2) &= \frac{2^2}{1-2}\\ &= \frac{4}{-1}\\ &= -4\end{aligned}$$

Test the slope of the curve on either side of $x=2$.

$$f'(1.9)=\frac{2(1.9)-(1.9)^2}{(1-1.9)^2}$$
$$=\frac{3.8-3.61}{0.81}$$
$$>0$$

Therefore, the curve rises to the left of $(2,-4)$.

$$f'(2.1)=\frac{2(2.1)-(2.1)^2}{(1-(2.1))^2}$$
$$=\frac{4.2-4.41}{1.1^2}$$
$$<0$$

Therefore, the curve falls to the right of $(2,-4)$.

Thus, the point $(2,-4)$ is a maximum.

Now, the curve has a vertical asymptote when $x=1$, since

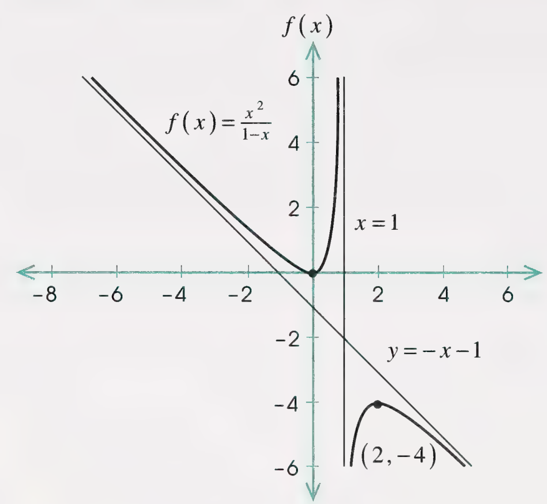
$f(x)=\frac{x^2}{1-x}$. (The denominator $\neq 0$.)

Also, since the degree of the numerator of the function is one more than the degree of the denominator, the curve has a slant asymptote. Use long division to determine the equation of the asymptote.

$$\begin{array}{r}
-x-1 \\
-x+1\overline{)\,x^2+0x+0} \\
\underline{x^2-1x} \\
x+0 \\
\underline{x-1} \\
1
\end{array}$$

Since $f(x)=-x-\frac{1}{1-x}$, the slant asymptote is $y=-x-1$.

Plot the extrema, and draw the asymptotes. Then sketch the graph of the function.



The maximum and minimum values are relative; there is no absolute minimum or maximum.

c. $f(x) = -x^2 - 2x$, where $-2 \le x \le 1$

$$f'(x) = -2x - 2$$

The stationary point occurs when $f'(x) = 0$.

$$\begin{aligned} -2x - 2 &= 0 \\ x &= -1 \end{aligned}$$

$$\begin{aligned} f(-1) &= -(-1)^2 - 2(-1) \\ &= 1 \end{aligned}$$

Test the slope of the curve on either side of $(-1, 1)$ to see if this point is a maximum or minimum.

$$\begin{aligned} f'(-1.1) &= -2(-1.1) - 2 \\ &= 0.2 \\ &> 0 \end{aligned}$$

Therefore, the curve rises on the left of $(-1, 1)$.

$$\begin{aligned} f'(-0.9) &= -2(-0.9) - 2 \\ &= -0.2 \end{aligned}$$

Therefore, the curve falls to the right of $(-1, 1)$.

Thus, the point $(-1, 1)$ is a relative maximum.

Now, because the function is defined on $[-2, 1]$, the ends of the interval must be tested.

$$\begin{aligned} f(-2) &= -(-2)^2 - 2(-2) \\ &= -4 + 4 \\ &= 0 \end{aligned}$$

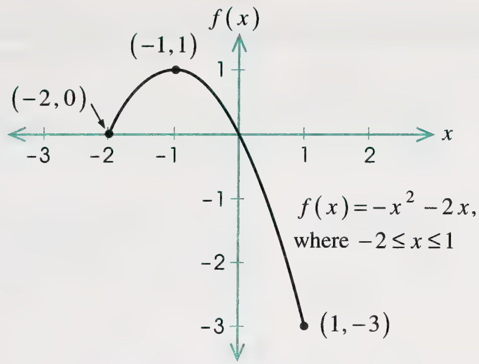
$$\begin{aligned} f'(-1.9) &= -2(-1.9) - 2 \\ &= 3.8 - 2 \\ &> 0 \end{aligned}$$

The graph rises to the right of $(-2, 0)$. Thus, $f(-2) = 0$ is a relative minimum.

$$\begin{aligned} f(1) &= -(1)^2 - 2(1) \\ &= -3 \end{aligned}$$

$$\begin{aligned} f'(0.9) &= -2(0.9) - 2 \\ &= -1.8 - 2 \\ &< 0 \end{aligned}$$

The graph falls to the left of $(1, -3)$. Thus, $f(1) = -3$ is a relative minimum.



From the graph, notice that $f(-1)=1$ is an absolute maximum and $f(1)=-3$ is an absolute minimum.

d. $y = x(2x-1)^{\frac{1}{2}}$

Differentiate using the product rule.

$$
\begin{aligned}
f'(x) &= x\frac{d}{dx}(2x-1)^{\frac{1}{2}} + (2x-1)^{\frac{1}{2}}\frac{d}{dx}(x) \\
&= x\left(\frac{1}{2}\right)(2x-1)^{\frac{1}{2}-1}(2) + (2x-1)^{\frac{1}{2}}(1) \\
&= x(2x-1)^{-\frac{1}{2}} + (2x-1)^{\frac{1}{2}} \\
&= (2x-1)^{-\frac{1}{2}}\left[x + (2x-1)\right] \\
&= (2x-1)^{-\frac{1}{2}}(3x-1)
\end{aligned}
$$

The stationary point would normally occur at the value of x for which $f'(x)=0$. However, if $f'(x)=0$, then $3x-1=0$ or $x=\frac{1}{3}$. This is impossible! If you replaced x by $\frac{1}{3}$, $f(x)$ would be undefined.

$$
\begin{aligned}
f\left(\frac{1}{3}\right) &= \left(\frac{1}{3}\right)\left[2\left(\frac{1}{3}\right)-1\right]^{\frac{1}{2}} \\
&= \left(\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{\frac{1}{2}}
\end{aligned}
$$

The square root of a negative number is non-real!

As a matter of fact, the domain of this function follows from $2x-1 \geq 0$. The domain is $\left[\frac{1}{2}, \infty\right)$.

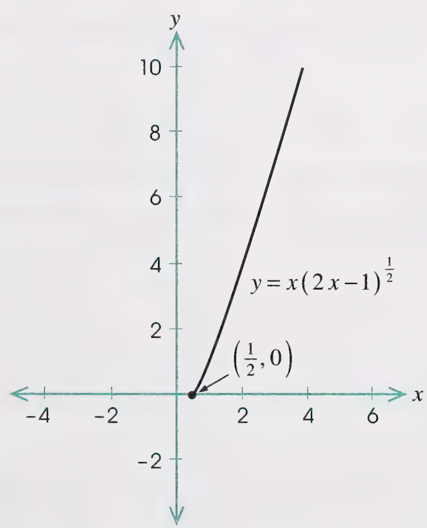
Test the end point.

$$
\begin{aligned}
f\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)\left[2\left(\frac{1}{2}\right)-1\right]^{\frac{1}{2}} \\
&= 0
\end{aligned}
$$

Test the slope of the curve on the right of point $\left(\frac{1}{2}, 0\right)$.

$$\begin{aligned} f'(1) &= \left(2(1)-1\right)^{-\frac{1}{2}}\left(3(1)-1\right) \\ &= 1 - \frac{1}{2}(2) \\ &= 2 \\ &> 0 \end{aligned}$$

The graph rises to the right of point $\left(\frac{1}{2}, 0\right)$; therefore, this point must be an absolute minimum.



Section 3: Activity 3

1. Determine the critical values of x from the derivative.

Since $f'(x) = 2x - 6$, only values for which $f'(x) = 0$ occur; therefore, there is no vertical tangent.

$$\begin{aligned} 2x - 6 &= 0 \\ x &= 3 \end{aligned}$$

The critical value of x is 3.

Find $f(3)$, and determine whether this value is a relative minimum, a relative maximum, or neither.

$$\begin{aligned} f(3) &= 3^2 - 6(3) - 3 \\ &= 9 - 18 - 3 \\ &= -12 \end{aligned}$$

Check the sign of the derivative on either side of $(3, -12)$.

Since $f'(x) = 2x - 6$, then $x > 3$ when $2x - 6 > 0$. Thus, the curve rises to the right of $(3, -12)$.

When $2x - 6 < 0$, then $x < 3$. Thus, the curve falls on the left of $(3, -12)$.

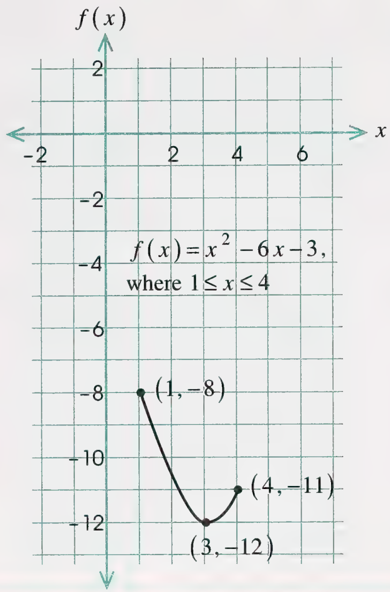
Therefore, the point $(3, -12)$ must be a relative minimum.

Now, since the function is defined on $1 \le x \le 4$, you must check the endpoints of the interval.

$$f(1) = 1^2 - 6(1) - 3 \qquad f(4) = 4^2 - 6(4) - 3$$
$$= -8 \qquad\qquad = -11$$

The point $(1, -8)$ is a relative maximum because the curve falls in the interval $[1, 3)$. The point $(4, -11)$ is a relative maximum because the curve rises in the interval $(3, 4]$.

Therefore, $f(3) = -12$ is an absolute minimum and $f(1) = -8$ is an absolute maximum. This fact is confirmed by the graph.



2. The critical values occur when $f'(x) = 0$.

Differentiate using the product rule.

$$f'(x) = uv' + vu'$$

$$u = (x-3)^3 \qquad v = (3x-2)^2$$
$$u' = 3(x-3)^2 \qquad v' = 6(3x-2)$$

$$\begin{aligned} f'(x) &= (x-3)^3(6)(3x-2) + (3x-2)^2(3)(x-3)^2 \\ &= 3(x-3)^2(3x-2)[2(x-3) + (3x-2)] \\ &= 3(x-3)^2(3x-2)[2x-6+3x-2] \\ &= 3(x-3)^2(3x-2)[5x-8] \end{aligned}$$

When $f'(x) = 0$, then $x - 3 = 0$ or $3x - 2 = 0$ or $5x - 8 = 0$. Thus, the critical values are $x = 3$, $x = \frac{2}{3}$, and $x = \frac{8}{5}$.

3. Stationary points occur when $f'(x) = 0$.

$$\begin{aligned} f'(x) = 5x^4 + 1 &= 0 \\ 5x^4 &= -1 \\ x^4 &= -\frac{1}{5} \end{aligned}$$

Impossible!

Since $5x^4 + 1 = 0$ has no real solutions, $f(x) = x^5 - x$ has no stationary points, and therefore, no relative maxima or minima.

4. $f(x) = -3x^{\frac{2}{3}}$

$$f'(x) = -3\left(\frac{2}{3}\right)x^{\frac{2}{3}-1}$$
$$= -2x^{-\frac{1}{3}}$$
$$= -\frac{2}{x^{\frac{1}{3}}}$$

When $x \to 0$, the slope becomes infinite.

Thus, $x = 0$ is a critical value.

When $x = 0$, $y = -3(0)^{\frac{2}{3}}$
$$= 0$$

Determine the slope of the curve on either side of point $(0,0)$.

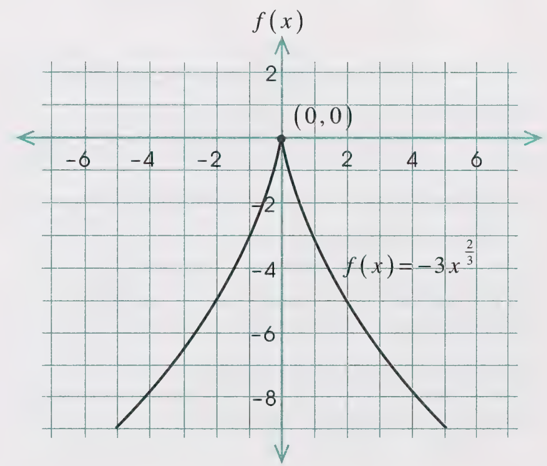
$$f'(-1) = -\frac{2}{(-1)^{\frac{1}{3}}}$$
$$= 2$$

Therefore, the curve rises on the left of point $(0,0)$.

$$f'(1) = -\frac{2}{(1)^{\frac{1}{3}}}$$
$$= -2$$

Therefore, the curve falls on the right of point $(0,0)$.

Thus, $f(x) = -3x^{\frac{2}{3}}$ has a relative maximum at $x = 0$.



5. The local maximum occurs when $f'(x) = 0$.

Now, $f'(x) = -4x + b$ and $f'(4) = 0$.

$$\therefore\ -4(4) + b = 0$$
$$b = 16$$

6. $f(x)=\frac{x^3+4}{x^2}$ may be written as $f(x)=x+\frac{4}{x^2}$

Using the second form of the function, you may notice that there is a vertical asymptote of $x=0$ and a slant asymptote of $y=x$.

$$\begin{aligned} f'(x)=1+4(-2)x^{-2-1} &= 0 \\ 1-8x^{-3} &= 0 \\ 1 &= 8x^{-3} \\ 1 &= \frac{8}{x^3} \\ x^3 &= 8 \\ x &= 2 \end{aligned}$$

Evaluate $f(2)$.

$$\begin{aligned} f(2) &= 2+\frac{4}{2^2} \\ &= 3 \end{aligned}$$

Determine the slope of the curve on either side of point $(2,3)$.

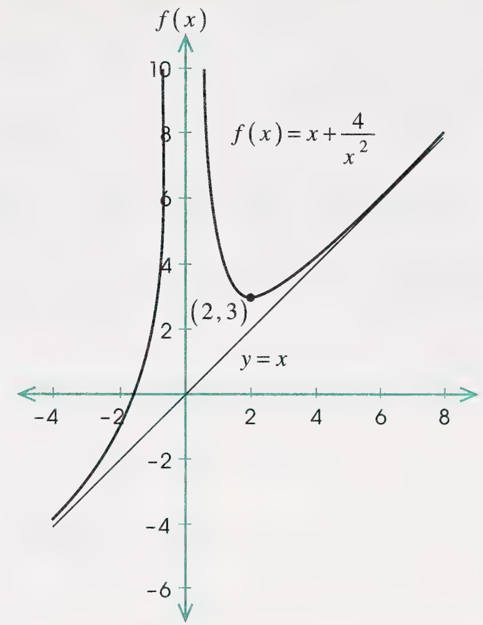
$$\begin{aligned} f'(1) &= 1-8(1)^{-3} \\ &= 1-8 \\ &= -7 \end{aligned}$$

Therefore, the curve falls on the left of point $(2,3)$.

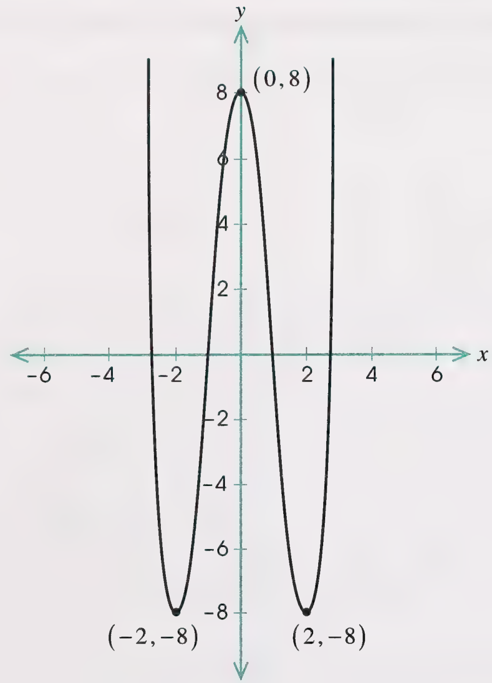
$$\begin{aligned} f'(1) &= 1-8(3)^{-3} \\ &= 1-\frac{8}{27} \\ &= \frac{19}{27} \end{aligned}$$

Therefore, the curve rises on the right of point $(2,3)$.

Thus, $(2,3)$ must be a relative minimum.



7.



8. Differentiate $y = [f(x)]^n$ using the chain rule.

$$y' = n[f(x)]^{n-1} f'(x)$$

Since (a, b) is a stationary point, $f'(a) = 0$.

$$\begin{aligned} \therefore\ y' &= n[f(a)]^{n-1} f'(a) \\ &= n[f(a)]^{n-1}(0) \\ &= 0 \end{aligned}$$

Therefore, (a, b) is a stationary point of $y = [f(x)]^n$.

Section 3: Follow-up Activities

Extra Help

1. $f(x) = -(x-2)(x+4)$
$= -x^2 - 2x + 8$

First, differentiate to find the stationary point.

$$f'(x) = -2x - 2$$

Let $f'(x)=0$.

$$-2x-2=0$$
$$x=-1$$

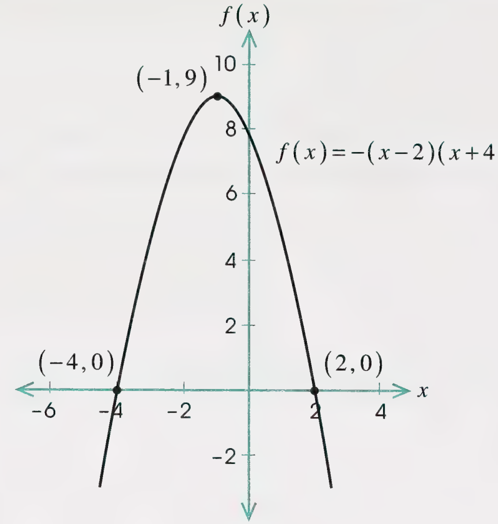
$$f(-1)=-(-1-2)(-1+4)$$
$$=-(-3)(3)$$
$$=9$$

The stationary point is $(-1, 9)$.

Next, decide where the function is rising and falling. Since the leading coefficient is negative, the graph falls on the interval $(-1, \infty)$. When $x<-1$, the derivative is positive; thus, the graph rises.

Now, find the x- and y-intercepts. The x-intercepts are 2 and -4; these values are obtained directly from the factors of $f(x)=-(x-2)(x+4)$. The y-intercept is $f(0)=-(0-2)(0+4)=8$.

Now, sketch the graph.



There is a maximum point at $(-1, 9)$. The graph rises for $x<-1$ and falls for $x>-1$.

2. First, differentiate to find the stationary points.

$$f(x)=2x^3-3x^2$$
$$f'(x)=6x^2-6x$$

Let $f'(x)=0$.

$$6x^2-6x=0$$
$$6x(x-1)=0$$

Therefore, the critical values of x are 0 and 1.

Now, $f(0)=0$ and $f(1)=2(1)^3-3(1)^2=-1$.

The stationary points are $(0,0)$ and $(1,-1)$.

Next, decide if the function is rising or falling when x increases without bound and also when x decreases without bound. Since the leading coefficient is positive, the graph rises when $x>1$. When $x<0$, the derivative is positive. For instance, $f'(-1)=6(-1)^2-6(-1)=12$. (The graph rises.)

Now, find the x- and y-intercepts.

Let $y=0$.

$$2x^3-3x^2=0$$
$$x^2(2x-3)=0$$
$$x=0 \text{ or } 2x-3=0$$
$$x=\frac{3}{2}$$

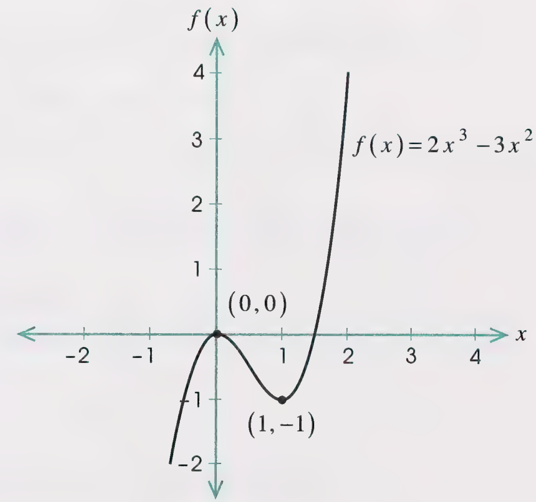
The x-intercepts are 0 and $\frac{3}{2}$.

Let $x=0$.

$$y=2(0)^3-3(0)^2$$
$$=0$$

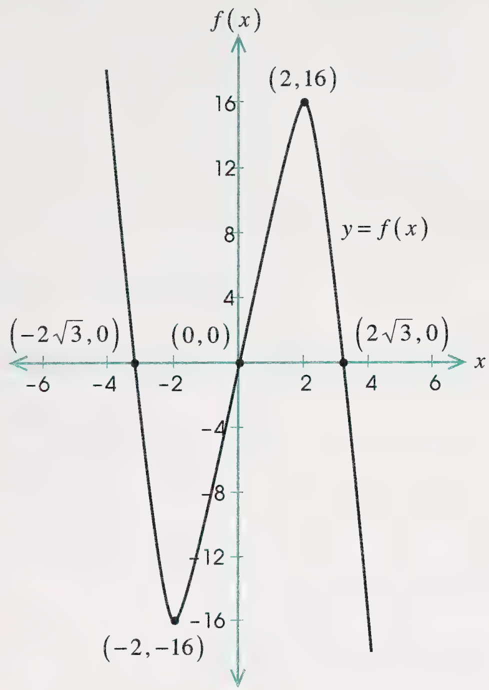
The y-intercept is 0.

Now, sketch the graph.



There is a maximum point at $(0,0)$ and a minimum at $(1,-1)$. The graph rises for $x<0$ or $x>1$; falls for $0<x<1$.

3. Since the leading coefficient is negative, the graph falls when $x > 2$. Plot the given points, and join them with a smooth curve.



Enrichment

1. First, find the slope of the chord.

$$f(2) = 2^3 \qquad f(-1) = (-1)^3$$
$$= 8 \qquad = -1$$

$$\begin{aligned}\text{slope of the chord} &= \frac{f(2) - f(-1)}{2 - (-1)} \\ &= \frac{8 - (-1)}{2 + 1} \\ &= \frac{9}{3} \\ &= 3\end{aligned}$$

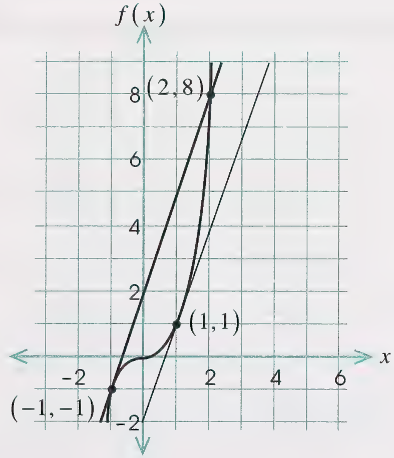
Next, find the slope of a tangent by differentiating the function.

$$f'(x) = 3x^2$$

If the tangent at $x = c$ is parallel to the chord, $f'(c) = \text{slope of}$ the chord.

$$\begin{aligned}3c^2 &= 3 \\ c^2 &= 1 \\ c &= \pm 1\end{aligned}$$

There is only one point in the interval $(-1, 2)$ where the tangent is parallel to the chord. That point is $(1, 1)$.



2. First, find the slope of the chord.

$$f(-1) = (-1)^{\frac{2}{3}} = 1 \qquad f(1) = (1)^{\frac{2}{3}} = 1$$

$$\begin{aligned}\text{slope of the chord} &= \frac{f(1) - f(-1)}{1-(-1)} \\ &= \frac{1-1}{1-(-1)} \\ &= \frac{0}{2} \\ &= 0\end{aligned}$$

Next, find the slope of a tangent by differentiating the function.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

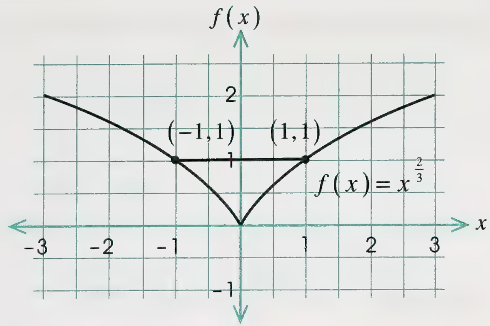
The derivative is undefined at $x = 0$. Since the function is not differentiable at $x = 0$, the conditions of the Mean Value Theorem are not met. However, try to find if there is a tangent parallel to the chord.

If the tangent at $x = c$ is parallel to the chord, $f'(c) =$ slope of the chord.

$$\begin{aligned}\frac{2}{3}c^{-\frac{1}{3}} &= 0 \\ c^{-\frac{1}{3}} &= 0 \\ c^{\frac{1}{3}} &= \frac{1}{0}\end{aligned}$$

c is undefined!

There is no point strictly between $x = -1$ and $x = 1$ where the tangent is horizontal. This result is true because the function is not differentaible at $x = 0$.



Section 4: Activity 1

1. a. $f(x) = -x^2 + 4x + 5$

Find the first and second derivatives.

$$f'(x) = -2x + 4$$
$$f''(x) = -2$$

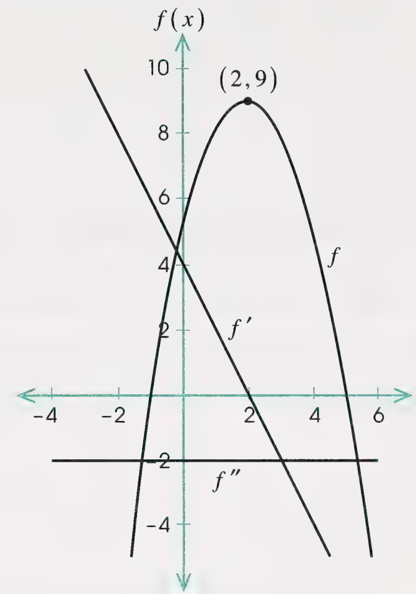
The curve is concave downward for all real values of x.

Next, find the stationary points which occur when $f'(x) = 0$.

$$-2x + 4 = 0$$
$$-2x = -4$$
$$x = 2$$

$$f(2) = -2^2 + 4(2) + 5$$
$$= -4 + 8 + 5$$
$$= 9$$

Since $(2, 9)$ lies on a curve which is concave downward, $(2, 9)$ must be a relative maximum.



b. $f(x) = x^3 - 3x - 4$

Find the first and second derivatives.

$$f'(x) = 3x^2 - 3$$
$$f''(x) = 6x$$

The curve is concave upward when $f''(x) > 0$.

$$6x > 0$$
$$x > 0$$

Thus, the curve is concave upward for the interval $(0, \infty)$.

The curve is concave downward when $f''(x) < 0$.

$$6x < 0$$
$$x < 0$$

Thus, the curve is concave upward for the interval $(-\infty, 0)$.

Next, find the stationary points which occur when $f'(x) = 0$.

$$3x^2 - 3 = 0$$
$$3x^2 = 3$$
$$x = \pm 1$$

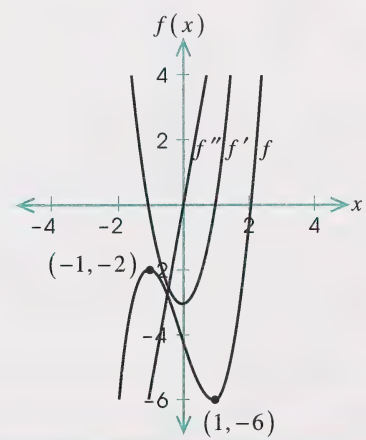
Now, $f(1) = 1^3 - 3(1) - 4$
$= 1 - 3 - 4$
$= -6$

Since $(1, -6)$ lies in the interval where the curve is concave upward, $(1, -6)$ must be a relative minimum.

$f(-1) = (-1)^3 - 3(-1) - 4$
$= -1 + 3 - 4$
$= -2$

Now, since $(-1, -2)$ lies in the interval where the curve is concave downward, $(-1, -2)$ must be a relative maximum.

The graph confirms these conclusions.



c. $f(x)=x+\frac{9}{x}$

Find the first and second derivatives.

$$f'(x)=1-9x^{2}$$
$$f''(x)=0-9(-2)\left(x^{-3}\right)$$
$$=\frac{18}{x^{3}}$$

The curve is concave upward when $f''(x)>0$. If $x>0$, then $x^{3}>0$ and $f''(x)=\frac{18}{x^{3}}>0$. Therefore, the curve is concave upward for the interval $(0,\infty)$.

The curve is concave downward when $f''(x)<0$. If $x<0$, then $x^{3}<0$ and $f''(x)=\frac{10}{x^{3}}<0$. Therefore, the curve is concave downward for the interval $(-\infty,0)$.

Next, find the stationary points which occur when $f'(x)=0$.

$$\begin{aligned} 1-9x^{-2} &= 0 \\ x^{-2} &= \frac{1}{9} \\ x^{2} &= 9 \\ x &= \pm 3 \end{aligned}$$

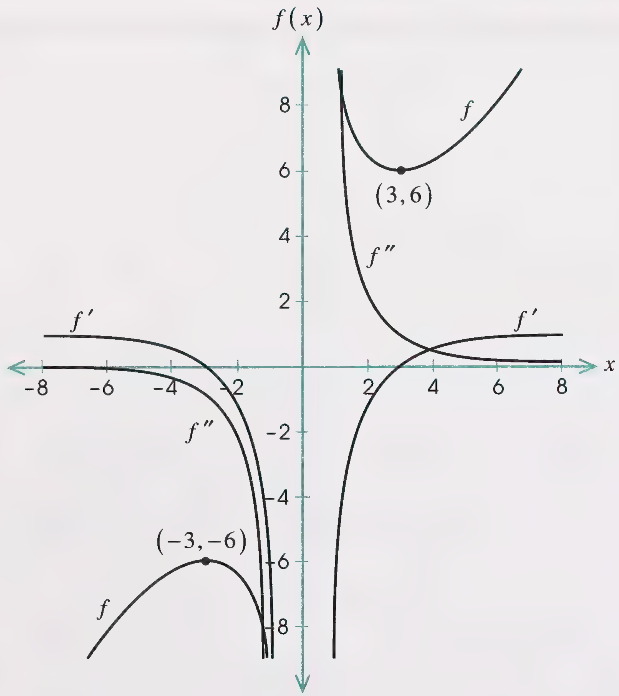
$$\begin{aligned} f(3) &= 3+\tfrac{9}{3} \\ &= 6 \end{aligned}$$

Since $(3,6)$ lies in the interval where the curve is concave upward, $(3,6)$ must be a relative minimum.

$$\begin{aligned} f(-3) &= -3+\frac{9}{-3} \\ &= -6 \end{aligned}$$

Now, since $(-3,-6)$ lies in the interval where the curve is concave downward, $(-3,-6)$ must be a relative maximum.

The graph confirms these conclusions.



d. $f(x)=(x-2)^4$

Find the first and second derivatives.

$$f'(x)=4(x-2)^3$$
$$f''(x)=12(x-2)^2$$

The curve is concave upward when $f''(x)>0$.

If $12(x-2)^2>0$, then x is any real value except 2. At $x=2$, the second derivative is 0.

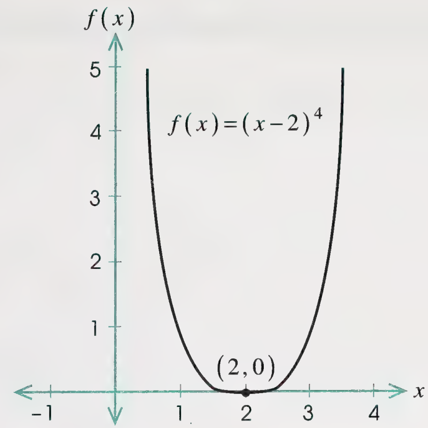
Next, find the stationary points which occur when $f'(x)=0$.

$$\begin{aligned} 4(x-2)^3 &= 0 \\ x &= 2 \end{aligned}$$

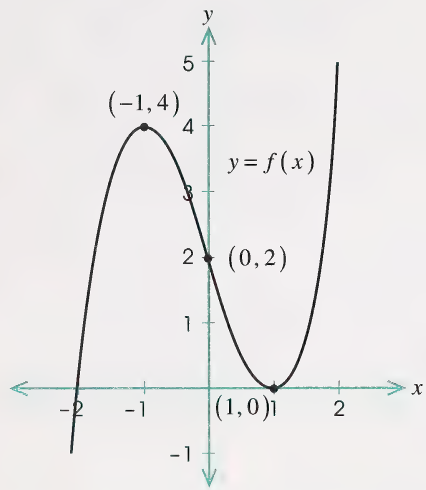
$$\begin{aligned} f(2) &= (2-2)^4 \\ &= 0 \end{aligned}$$

Now, the second derivative at $x=2$ is 0. The curve is neither concave upward or downward at this point. However, both to the right and to the left, $f(x)=(x-2)^4>0$. Therefore, $(2,0)$ must be a relative minimum.

The graph confirms these conclusions.



2.



3. **a.** $f(x) = x^2 - 6x + 2$

$f'(x) = 2x - 6$

Locate the stationary points.

$$f'(x) = 2x - 6 = 0$$
$$x = 3$$

$$f(3) = 3^2 - 6(3) + 2$$
$$= 9 - 18 + 2$$
$$= -7$$

Next, find the values of the second derivative at this point.

$$f''(x) = 2$$

Since the second derivative at $x = 3$ is positive, $f(3) = -7$ is a relative minimum.

b. $f(x) = -(x+2)^4 + 1$

$f'(x) = -4(x+2)^3$

Locate the stationary points.

$$f'(x) = -4(x+2)^3 = 0$$
$$x = -2$$

$$
\begin{aligned}
f(-2) &= -(-2+2)^4 + 1 \\
&= -0+1 \\
&= 1
\end{aligned}
$$

Next, find the values of the second derivative at this point.

$$
\begin{aligned}
f''(x) &= -12(x+2)^2 \\
f''(-2) &= -12(-2+2)^2 \\
&= 0
\end{aligned}
$$

The Second Derivative Test fails.

As an alternative, test the slope on either side of $x=-2$.

$$
\begin{aligned}
f'(-3) &= -4(-3+2)^3 \\
&= -4(-1) \\
&= 4
\end{aligned}
$$

On the left, the function increases.

$$
\begin{aligned}
f'(-1) &= -4(-1+2)^3 \\
&= -4(1) \\
&= -4
\end{aligned}
$$

On the right, the function decreases.

Since the graph rises and then falls, the point $(-2,1)$ is a maximum point.

$f(-2)=1$ is the maximum value of the function.

c. $$
\begin{aligned}
y &= \frac{x^2}{x^2+4} \\
y' &= \frac{\left(x^2+4\right)\frac{d}{dx}\left(x^2\right) - x^2\frac{d}{dx}\left(x^2+4\right)}{\left(x^2+4\right)^2} \\
&= \frac{\left(x^2+4\right)(2x) - x^2(2x)}{\left(x^2+4\right)^2} \\
&= \frac{2x\left(x^2+4-x^2\right)}{\left(x^2+4\right)^2} \\
&= \frac{8x}{\left(x^2+4\right)^2}
\end{aligned}
$$

Locate the stationary points.

$$
\begin{aligned}
y' = \frac{8x}{\left(x^2+4\right)^2} &= 0 \\
x &= 0
\end{aligned}
$$

Now, when $x=0$, $y=0$.

Next, find the second derivative.

$$y'' = \frac{\left(x^2+4\right)^2 \frac{d}{dx}(8x) - 8x\frac{d}{dx}\left(x^2+4\right)^2}{\left(x^2+4\right)^4}$$

$$= \frac{\left(x^2+4\right)^2 (8) - 8x(2)\left(x^2+4\right)(2x)}{\left(x^2+4\right)^4}$$

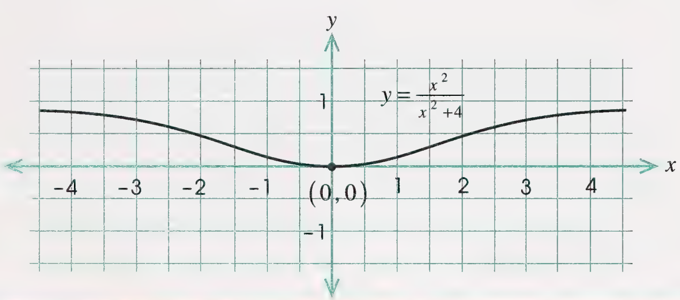
$$= \frac{8\left(x^2+4\right)\left[\left(x^2+4\right)-4x^2\right]}{\left(x^2+4\right)^4}$$

$$= \frac{8\left(x^2+4\right)\left[-3x^2+4\right]}{\left(x^2+4\right)^4}$$

At $x=0$, $y'' = \dfrac{8(4)(4)}{4^4}$

> 0

Therefore, the point $(0,0)$ is a relative minimum.



d. $y = \sqrt{x}\,(x-2)$ is defined for $x \geq 0$.

You must test the function at $x = 0$, the endpoint of the interval on which the function is defined, for an extreme value.

When $x = 1$, $y = \sqrt{1}(1-2) = -1$. Since this value of the function is less than at $(0,0)$, a local maximum must be located at the origin.

$$y = \sqrt{x}\,(x-2)$$
$$= x^{\frac{1}{2}}(x-2)$$
$$= x^{\frac{3}{2}} - 2x^{\frac{1}{2}}$$

$$y' = \frac{3}{2}x^{\frac{1}{2}} - \frac{2(1)}{2}x^{-\frac{1}{2}}$$
$$= \frac{1}{2}x^{-\frac{1}{2}}(3x-2)$$

Locate the stationary points.

$$y' = \frac{1}{2}x^{-\frac{1}{2}}(3x-2) = 0$$

$\therefore\ 3x - 2 = 0$

$x = \dfrac{2}{3}$

When $x = \frac{2}{3}$, $y = \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{2}{3} - 2\right)$

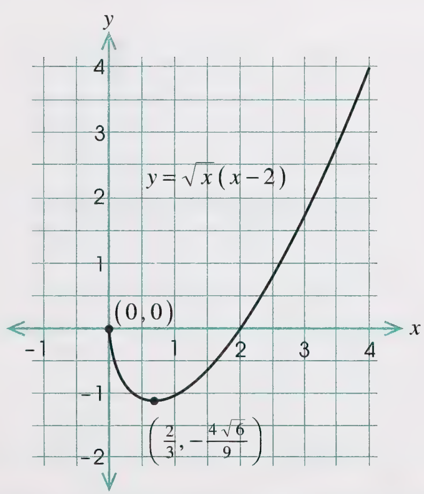
$$= \frac{-4\sqrt{2}\sqrt{3}}{3\sqrt{3}\sqrt{3}}$$

$$= -\frac{4\sqrt{6}}{9}$$

Use the Second Derivative Test.

$y'' = \frac{3}{4}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$, which is positive whenever $x > 0$.

Therefore, the point $\left(\frac{2}{3}, -\frac{4\sqrt{6}}{9}\right)$ is a relative minimum.



Section 4: Activity 2

1. a. This is a polynomial function. Since its derivatives are finite for all x, the only situation for which points of inflection may occur is when $f''(x) = 0$.

Differentiate.

$$f'(x) = 3(x-1)^2$$
$$f''(x) = 6(x-1)$$

When $f''(x) = 0$, $x = 1$.

$$f(1) = (1-1)^3 - 2$$
$$= -2$$

Thus, the point of inflection may occur at $(1, -2)$.

Be sure to confirm this by checking the concavity of the curve on either side of this point.

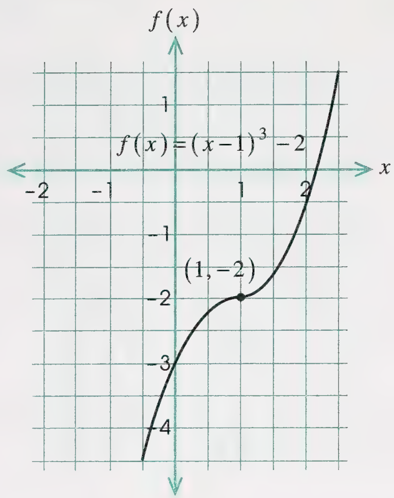
$$f''(0) = 6(0-1)$$
$$= -6$$
$$< 0$$

On the left, the curve is concave downward.

$$\begin{aligned} f''(2) &= 6(2-1) \\ &= 6 \\ &> 0 \end{aligned}$$

On the right, the curve is concave upward.

Since the curve changes concavity at $x = 1$, then $(1, -2)$ is a point of inflection.



b. Again, this is a polynomial function. As in question a., its derivatives are finite for all *x*. The only situation for which points of inflection may occur is when $f''(x) = 0$.

Differentiate.

$$f(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - x^2 + x - 1$$

$$f'(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$$

$$f''(x) = x^2 - x - 2$$

Let $f''(x) = 0$.

$$\begin{aligned} x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x - 2 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = 2 \quad &\quad\;\; x = -1 \end{aligned}$$

$$\begin{aligned} f(-1) &= \frac{1}{12}(-1)^4 - \frac{1}{6}(-1)^3 - (-1)^2 + (-1) - 1 \\ &= -\frac{33}{12} \\ &= -\frac{11}{4} \end{aligned}$$

$$\begin{aligned} f(2) &= \frac{1}{12}(2)^4 - \frac{1}{6}(2)^3 - (2)^2 + (2) - 1, \\ &= -3 \end{aligned}$$

Therefore, the points of inflection may occur at $\left(-1, -2\frac{3}{4}\right)$ and $(2, -3)$.

Confirm that $\left(-1, -2\frac{3}{4}\right)$ is a point of inflection by checking the concavity of the curve on either side of the point.

$$\begin{aligned} f''(-2) &= (-2)^2 - (-2) - 2 \\ &= 4 \\ &> 0 \end{aligned}$$

On the left of $x = -1$, the curve is concave upward.

$$\begin{aligned} f''(0) &= 0^2 - 0 - 2 \\ &= -2 \\ &< 0 \end{aligned}$$

On the right of $x = -1$, the curve is concave downward.

Thus, the point $\left(-1, -2\frac{3}{4}\right)$ is a point of inflection.

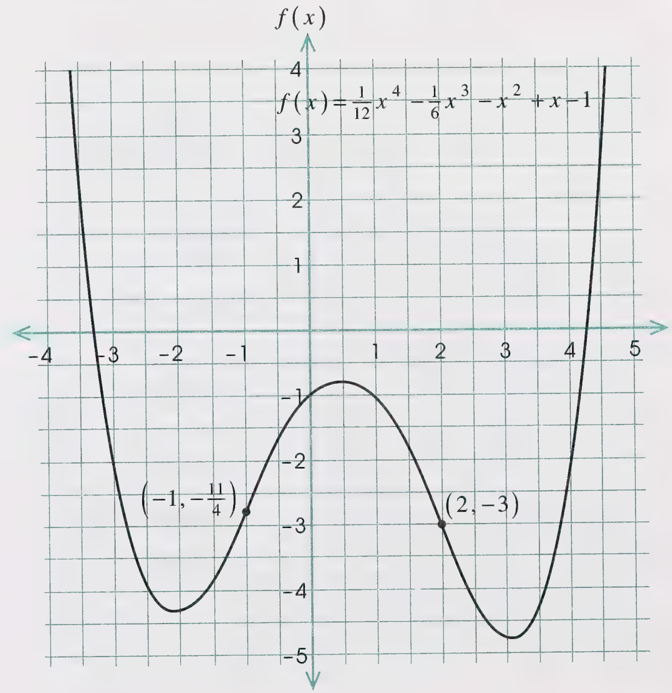
Now, confirm that $(2, -3)$ is a point of inflection.

$$\begin{aligned} f''(3) &= (3)^2 - (3) - 2 \\ &= 4 \\ &> 0 \end{aligned}$$

On the right of $x = 2$, the curve is concave upward.

You also know that because $f''(0) < 0$, the curve is concave downward on the left of $x = 2$.

Since the curve changes concavity at $x = 2$, then $(2, -3)$ is a point of inflection.



c. $f(x) = -(x-2)^{\frac{1}{5}} + 3$

$$\begin{aligned} f'(x) &= -\frac{1}{5}(x-2)^{-\frac{4}{5}} + 0 \\ &= \frac{-1}{5\left[(x-2)^{\frac{1}{5}}\right]^4} \end{aligned}$$

The first derivative is positive for all x except for $x=2$. As $x \to 2$, $f'(x)$ becomes infinite. A point of inflection may occur if $f''(x)$ becomes infinite at $x=2$ as well.

Now, $f''(x)=\frac{4}{25}(x-2)^{-\frac{9}{5}}$ and $f''(x)$ becomes infinite as $x \to 2$.

$$\begin{aligned} f(2) &= -(2-2)^{\frac{1}{5}}+3 \\ &= 3 \end{aligned}$$

Therefore, the point of inflection may be $(2, 3)$.

Check concavity on either side of $x=2$.

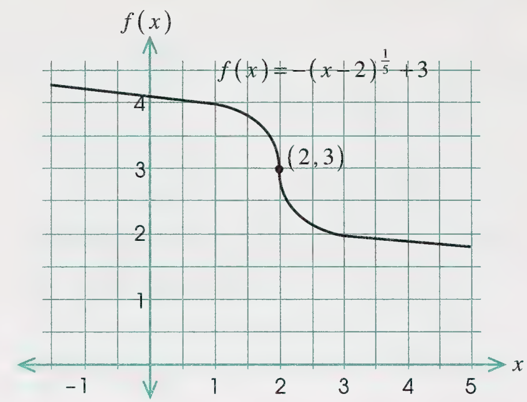
$$\begin{aligned} f''(1) &= \frac{4}{5}(-1)^{-\frac{9}{5}} \\ &= -\frac{4}{5} \\ &< 0 \end{aligned}$$

The curve is concave downward on the left.

$$\begin{aligned} f''(3) &= \frac{4}{5}(1)^{-\frac{9}{5}} \\ &= \frac{4}{5} \\ &> 0 \end{aligned}$$

The curve is concave upward on the right.

Thus, the point $(2, 3)$ is a point of inflection.

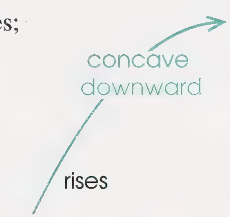


2. There is no inflection point. Even though $y''=0$ at $x=0$, the curve is concave downward on either side of $(0, 2)$. The point $(0, 2)$ is a maximum point.

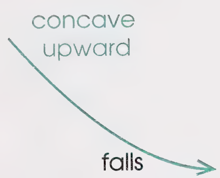
Section 4: Follow-up Activities

Extra Help

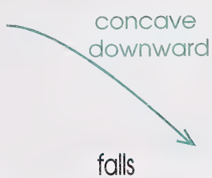
1. **a.** Since $f'(x)>0$, the graph rises; since $f''(x)<0$, the graph is concave downward.



b. Since $f'(x)<0$, the graph falls; since $f''(x)>0$, the graph is concave upward.



c. Since $f'(x)<0$ the graph falls; since $f''(x)<0$, the graph is concave downward.



2. The portion for which $f''(x)>0$ is a semicircle, with endpoints $(-2,0)$ and $(2,0)$. It passes through point $(0,-2)$.

Enrichment

1. a. $f'(x)=3x^2$

$f''(x)=6x$

Evaluate the derivatives at $(1,1)$.

$$f'(1)=3(1)^2 \qquad f''(1)=6(1)$$
$$=3 \qquad\qquad =6$$

$$\therefore\ r=\frac{\left[1+\left(f'(1)\right)^2\right]^{\frac{3}{2}}}{|f''(1)|}$$
$$=\frac{\left[1+3^2\right]^{\frac{3}{2}}}{6}$$
$$=\frac{10^{\frac{3}{2}}}{6}$$
$$=\frac{10\sqrt{10}}{6}$$
$$=\frac{5\sqrt{10}}{3}$$

b. $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}$

Evaluate the derivatives at $(4, 2)$.

$$f'(4) = \frac{1}{2}(4)^{-\frac{1}{2}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f''(4) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4\left(\sqrt{4}\right)^3} = -\frac{1}{32}$$

$$\therefore\ r = \frac{\left[1+\left(f'(4)\right)^2\right]^{\frac{3}{2}}}{\left|f''(4)\right|} = \frac{\left[1+\left(\frac{1}{4}\right)^2\right]^{\frac{3}{2}}}{\frac{1}{32}} = \frac{\left[1+\frac{1}{16}\right]^{\frac{3}{2}}}{\frac{1}{32}} = 32\left[\frac{17}{16}\right]^{\frac{3}{2}} = \frac{32\left(17\sqrt{17}\right)}{64} = \frac{17\sqrt{17}}{2}$$

2. The radius of curvature is 2 (simply the radius of the circle itself).

Section 5: Activity 1

1. **a.** $f(x) = \dfrac{x^2+1}{x^2-1}$

Domain: The domain is $\{x \mid x \neq \pm 1\}$ as the denominator, $x^2 - 1$, must be non-zero.

Intercepts: If $f(x) = 0$, then $x^2 + 1 = 0$. Impossible!

Therefore, there are no x-intercepts.

Since $f(0) = -1$, the graph crosses the y-axis at $(0, -1)$.

Asympototes: $$\lim_{x\to\infty}\frac{x^2+1}{x^2-1} = \lim_{x\to\infty}\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(1-\frac{1}{x^2}\right)} = \frac{1+0}{1-0} = 1$$

The horizontal asymptote is $y = 1$.

The vertical asymptotes occur when the denominator $x^2 - 1 = 0$, or when $x = \pm 1$.

Symmetries: Since the equation would not be changed when (x, y) is replaced by $(-x, y)$, the curve is symmetrical about the *y*-axis.

$$f(x)=\frac{x^2+1}{x^2-1}=\frac{(-x)^2+1}{(-x)^2-1}$$

Intervals of increase and decrease: $f(x)=\frac{x^2+1}{x^2-1}$

Differentiate using the quotient rule.

$$f'(x)=\frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}$$

$$=\frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2}$$

$$=\frac{2x\left(x^2-1-x^2-1\right)}{\left(x^2-1\right)^2}$$

$$=\frac{-4x}{\left(x^2-1\right)^2}$$

The sign of the first derivative depends on the numerator $(-4x)$.

If $f'(x)>0$, then $-4x>0$ or $x<0$.

The function rises for $x<-1$ and $-1<x<0$. Remember, $x\neq-1$.

If $f'(x)<0$, then $-4x<0$ or $x>0$.

The function falls for $0<x<1$ and $x>1$. Remember, $x\neq 1$.

Extrema: The stationary point occurs at $(0,1)$ since $f'(0)=0$. This is a relative maximum. On its left, the curve rises; on its right, the curve falls.

Concavity: Find the second derivative using the product rule.

$$f'(x)=-4x\left(x^2-1\right)^{-2}$$

$$f''(x)=-4x\frac{d}{dx}\left(x^2-1\right)^{-2}+\left(x^2-1\right)^{-2}\frac{d}{dx}(-4x)$$

$$=-4x(-2)\left(x^2-1\right)^{-3}(2x)+\left(x^2-1\right)^{-2}(-4)$$

$$=-4\left(x^2-1\right)^{-3}\left[-4x^2+\left(x^2-1\right)\right]$$

$$=-4\left(x^2-1\right)^{-3}\left(-3x^2-1\right)$$

$$=\frac{4\left(3x^2+1\right)}{\left(x^2-1\right)^3}$$

Since the numerator of $f''(x)$ is always positive, the sign of $f''(x)$ will be the same as the sign of $x^2 - 1$.

Now, when $f''(x) > 0$, $x^2 - 1 > 0$

$$x^2 > 1$$

$x > 1$ or $x < -1$

The curve is concave upward in the interval $(-\infty, -1) \cup (1, \infty)$.

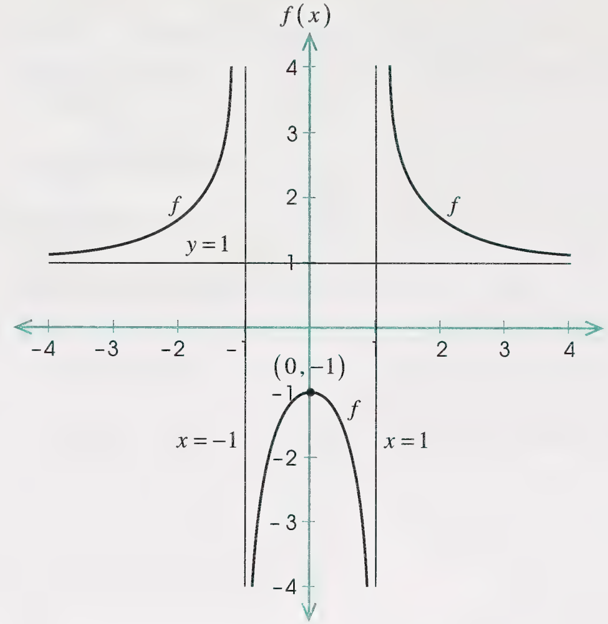
Now, when $f''(x) < 0$, $x^2 - 1 < 0$

$$x^2 < 1$$
$$-1 < x < 1$$

The curve is concave downward in the interval $(-1, 1)$.

Points of inflection: There are no inflection points since concavity changes at $x = \pm 1$ (values of x for which the function is undefined).

Sketch: The graph is as follows:



b. $f(x) = \left(x^2 - 4x\right)^{\frac{1}{2}}$

Domain: The function is defined for $x^2 - 4x \geq 0$

$$x(x-4) \geq 0$$

When the product of the factors is 0, then $x = 0$ or $x = 4$.

When the product of the factors is positive, either both factors are positive or both negative. Both are positive when $x > 4$; both are negative when $x < 0$. Therefore, the domain is $(-\infty, 0] \cup [4, \infty)$.

Intercepts: If $f(x) = 0$, then $x = 0$ or $x = 4$. The graph crosses the *x*-axis at $(0, 0)$ and $(4, 0)$. The *y*-intercept is 0.

Asymptotes: none

Symmetries: none

Intervals of increase and decrease

$$f(x) = \left(x^2 - 4x\right)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}\left(x^2 - 4x\right)^{-\frac{1}{2}}(2x - 4)$$

$$= \frac{x-2}{\left(x^2 - 4x\right)^{\frac{1}{2}}}$$

The sign of the first derivative depends on the sign of $x - 2$.

When $f'(x) > 0$, then $x - 2 > 0$ or $x > 2$. But the function is undefined for $0 < x < 4$. Thus, the function rises when $x > 4$.

When $f'(x) < 0$, then $x - 2 < 0$ or $x < 2$. But the function is undefined for $0 < x < 4$. Thus, the function falls when $x < 0$.

Extrema: Test the endpoints $(0, 0)$ and $(4, 0)$ of the domain of the function. Since the function rises when $x > 4$, $(4, 0)$ must be a relative minimum. Since the function falls when $x < 0$, $(0, 0)$ is also a relative minimum.

Concavity: Find the second derivative using the product rule.

$$f'(x) = \frac{x-2}{\left(x^2 - 4x\right)^{\frac{1}{2}}}$$

$$= (x-2)\left(x^2 - 4x\right)^{-\frac{1}{2}}$$

$$f''(x)$$

$$= \left(x^2 - 4x\right)^{-\frac{1}{2}} \frac{d}{dx}(x-2) + (x-2)\frac{d}{dx}\left(x^2 - 4x\right)^{-\frac{1}{2}}$$

$$= \left(x^2 - 4x\right)^{-\frac{1}{2}}(1) + (x-2)\left(\frac{-1}{2}\right)\left(x^2 - 4x\right)^{-\frac{3}{2}}(2x-4)$$

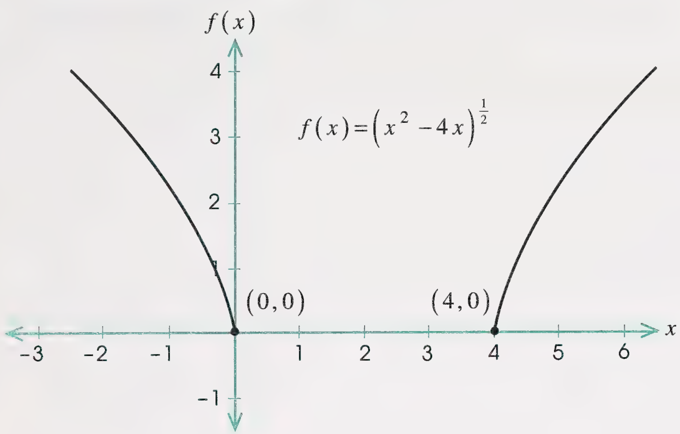
$$= \left(x^2 - 4x\right)^{-\frac{3}{2}}\left[\left(x^2 - 4x\right) - (x-2)(x-2)\right]$$

$$= \left(x^2 - 4x\right)^{-\frac{3}{2}}\left[x^2 - 4x - \left(x^2 - 4x + 4\right)\right]$$

$$= \frac{-4}{\left(x^2 - 4x\right)^{\frac{3}{2}}}$$

Now, $f''(x)<0$ for $x<0$ and $x>4$. The curve is concave downward throughout its domain except, of course, at the endpoints $(0,0)$ and $(4,0)$, where the slope is undefined.

Sketch: The graph is as follows:



2. $y = 2\sqrt{x} + \frac{1}{x}$

Domain: The function is defined for $x>0$. x must be non-zero since $\frac{1}{x}$ would be undefined.

Intercepts: There is no *y*-intercept since $f(0)$ is undefined.

When $y=0$, then $2\sqrt{x}+\frac{1}{x}=0$. But this equation has no solution as both terms on the left side are positve. There is no *x*-intercept.

Asymptotes: The vertical asymptote is $x=0$.

Symmetries: none

Intervals of increase and decrease

$$\begin{aligned} y &= 2\sqrt{x} + \frac{1}{x} \\ &= 2\left(x^{\frac{1}{2}}\right) + x^{-1} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}} - 1x^{-2} \\ &= \frac{1}{x^{\frac{1}{2}}} - \frac{1}{x^2} \end{aligned}$$

When $\frac{dy}{dx} > 0$, then $\frac{1}{x^{\frac{1}{2}}} - \frac{1}{x^2} > 0$

$$
\begin{aligned}
\frac{1}{x^{\frac{1}{2}}} &> \frac{1}{x^2} \\
\frac{1}{x} &> \frac{1}{x^4} \qquad \text{(Square both sides.)} \\
\frac{x^4}{x} &> 1 \\
x^3 &> 1 \\
x &> 1
\end{aligned}
$$

The curve rises for $x > 1$.

Similarly, when $\frac{dy}{dx} < 0$, $x < 1$. Since the original function is only defined when $x > 0$, the interval of decrease is $0 < x < 1$.

Note: The stationary point occurs when $x = 1$.

When $x = 1$, $y = 2\sqrt{1} + \frac{1}{1}$

$$= 3$$

Extrema: Since the curve falls to the left of $(1, 3)$ and rises to the right of $(1, 3)$, then $(1, 3)$ is a relative minimum.

Concavity: Recall $\frac{dy}{dx} = x^{-\frac{1}{2}} - 1x^{-2}$.

$$\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}} + 2x^{-3}$$

When $y'' > 0$, $-\frac{1}{2}x^{-\frac{3}{2}} + 2x^{-3} > 0$

$$
\begin{aligned}
\frac{2}{x^3} &> \frac{1}{2x^{\frac{3}{2}}} \\
\frac{4}{x^6} &> \frac{1}{4x^3} \\
16 &> \frac{x^6}{x^3} \\
x^3 &< 16 \\
x &< \sqrt[3]{2^4} \\
&< \sqrt[3]{2^3 \bullet 2} \\
&< 2\sqrt[3]{2}
\end{aligned}
$$

Square both sides.

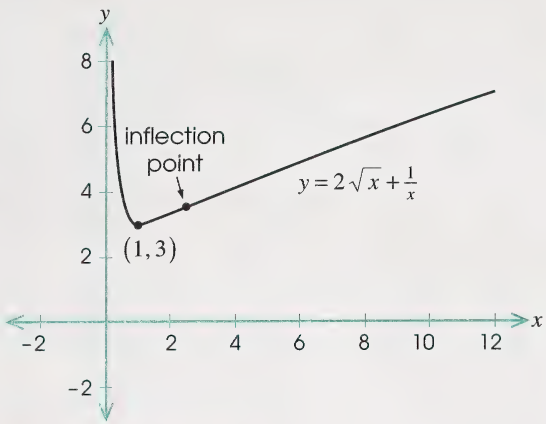
The cross products are the same. You can cross multiply as all terms are +.

The curve is concave upward in the interval $(0, 2\sqrt[3]{2})$.

Similarly, the curve is concave downward for the interval $(2\sqrt[3]{2}, \infty)$.

Inflection Point: The change of concavity occurs at $x = 2\sqrt[3]{2}$. The approximate location of this point is $(2.52, 3.57)$.

Sketch: The graph is as follows:



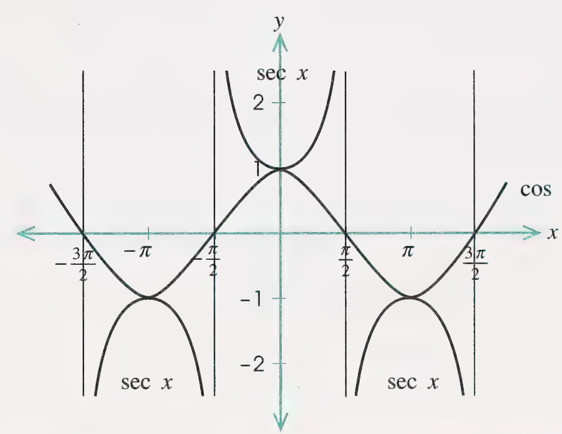
Section 5: Activity 2

1. Since $y' = \cos x$, the slope at $(0, 0)$ is $\cos 0 = 1$. This is a maximum slope because the graph of the derivative, $y' = \cos x$, ranges between ± 1.

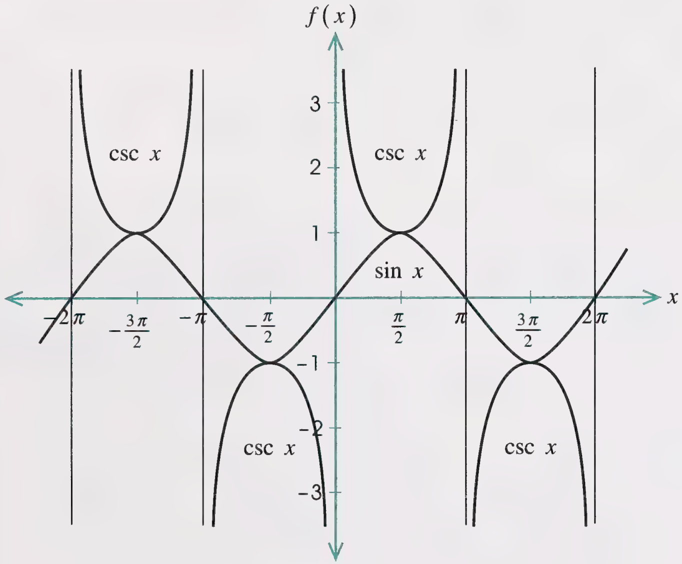
2. The number of daylight hours is increasing the fastest when y' is a maximum. Now, $y' = 4\left(\frac{2\pi}{365}\right)\cos\left[\frac{2\pi(x-80)}{365}\right]$. Since $\cos 0 = 1$ is a maximum value of the cosine function, in this example, the maximum will occur when $x = 80$. ($x - 80$ will be 0 for that value of x.) The 80th day of the year is March 21, when night and day are equal in length.

3. To sketch $y = \sec x$ for $-\frac{3\pi}{2} < x < \frac{3\pi}{2}$, first sketch $y = \cos x$; then take the reciprocal of the y-value of each point, since $\sec x = \frac{1}{\cos x}$ $\left(\text{e.g. } \cos \frac{\pi}{3} = \frac{1}{2}\right)$. Therefore, $\sec \frac{\pi}{3} = 2$. Now as $x \to \frac{\pi}{2} + n\pi$, where $n \in I$, $\cos x \to 0$. The graph of $f(x) = \sec x$ has vertical asymptotes at $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$ for this particular question. Since $\sec x = \frac{1}{\cos x}$, the maximum values of $\cos x$ will yield minimum values of $\sec x$; minimum values of $\cos x$ will yield maximum values of $\sec x$. As in the diagram, the minimum value of $\sec x$ is -1 at $x = 0$; the maximum is 1 at $x = -\pi$ and $x = \pi$. The stationary points occur at $f'(x) = 0$.

   Since $f'(x) = \sec x \tan x$ and $\sec x$ is never 0, then $\tan x = 0$ at the stationary points. Therefore, $x = -\pi$, $x = 0$, and $x = \pi$.



4. The stationary points occur when $f'(x)=0$. Since $f(x)=\csc x$, $f'(x)=-\csc x \cot x$. As $\csc x \neq 0$ for all real values of x, if $f'(x)=0$, then $\cot x=0$. When $\cot x=0$, $x=\frac{\pi}{2}+n\pi$, where $n \in I$.



5. Stationary points occur when $f'(x)=0$.

$$f(x)=x\sin x$$

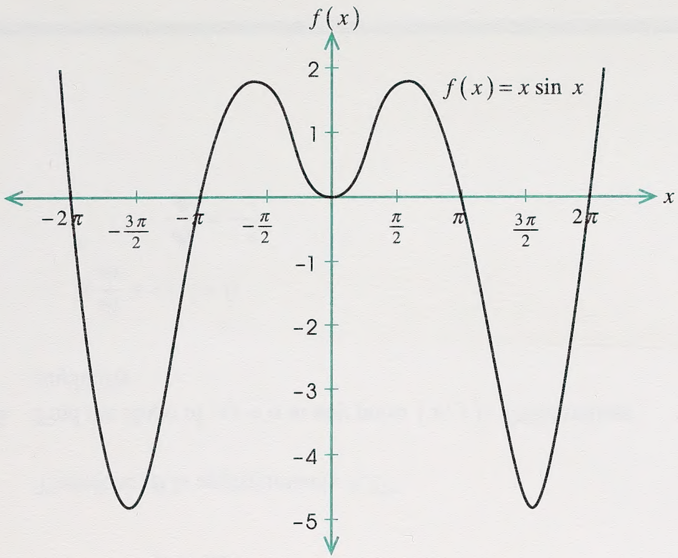
$$\begin{aligned} f'(x) &= x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x) \\ &= x\cos x+\sin x \end{aligned}$$

$$\begin{aligned} f'(0) &= 0\cos 0+\sin 0 \\ &= 0(1)+0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f''(x) &= x\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(x)+\cos x \\ &= -x\sin x+\cos x+\cos x \\ &= -x\sin x+2\cos x \end{aligned}$$

$$\begin{aligned} f''(0) &= -0\sin 0+2\cos 0 \\ &= 0+2(1) \\ &= 2 \\ &> 0 \end{aligned}$$

By using the Second Derivative Test, a local minimum occurs at $(0,0)$.

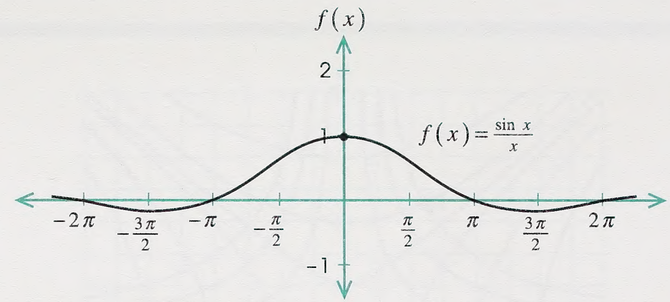


6. Since $f(x)=\frac{\sin x}{x}$, $f'(x)=\frac{x\cos x-\sin x}{x^2}$

Use your calculator to find $f'(-1)$. Make certain your calculator is set on radian mode.

$$\begin{aligned} f'(-1) &= \frac{-\cos(-1)-\sin(-1)}{(-1)^2} \\ &= 1.30116\ldots \\ &> 0 \end{aligned}$$

The graph rises at $x=-1$.



Enrichment

1. Find the point of intersection of $y=4x$ and $y=x^3$.

$$\begin{aligned} 4x &= x^3 \\ x^3-4x &= 0 \\ x\left(x^2-4\right) &= 0 \\ x(x+2)(x-2) &= 0 \end{aligned}$$

$x=0$ or $x+2=0$ or $x-2=0$

$x=-2$ $\quad$ $x=2$

When $x=2$, $y=4(2)$
$=8$

The point of intersection in the first quadrant is $(2,8)$.

Find the slopes of the tangents.

Let m_1 be the slope of $y = 4x$. Since $y' = 4$, $m_1 = 4$.

Let m_2 be the slope of $y = x^3$. Since $y' = 3x^2$,

$m_2 = 3(2^2) = 12.$

$$\therefore \ \tan \beta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\tan \beta = \frac{12 - 4}{1 + 12(4)}$$

$$\tan \beta = \frac{8}{49}$$

$$\beta \doteq 9.27$$

Therefore, β is approximately 9.27°.

2. Find the slope of $xy = a$ at any point (x, y). Differentiate implicitly.

$$x\frac{dy}{dx} + y(1) = 0$$

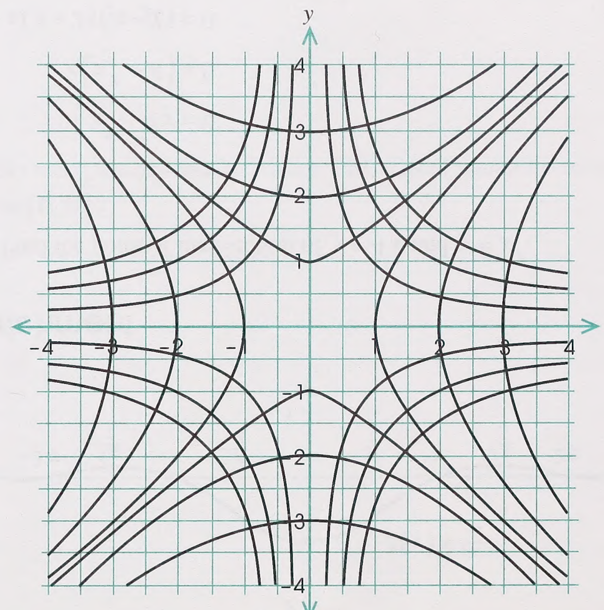
$$\frac{dy}{dx} = -\frac{y}{x}$$

Find the slope of $x^2 - y^2 = b$ at any point (x, y). Differentiate implicitly.

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

If (x, y) is a point of intersection, then the product of the slopes at that point is $\left(-\frac{y}{x}\right)\left(\frac{x}{y}\right) = -1$. The curves are perpendicular.



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